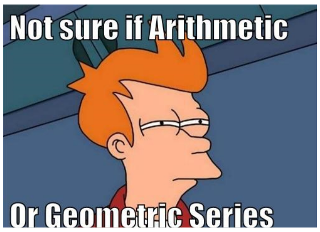
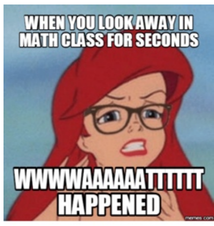
Sequences & Series: Arithmetic & Geometric





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This is a long worksheet to cater for students that want extra practice. If you want a shortcut, but still be sure to cover one of each type then follow the pink highlighted questions.

- **Higher level students** should be able to do all questions to be sure to get a 7
- **Standard level students** should be able to do questions 1-56 and 63-77 to be sure to get a grade 7

The sections 'with … ' should only be done once a student has already covered those topics in their course.

Note: There is no sigma notation or recurrence relations in this sheet. These topics are on a separate worksheet.

Quick Reminders

Notation and Definitions

n	u_n	S_n
n = term number This is always a whole number since it represents a certain term. For example: $n = 1$ means look for 1st term $n = 3$ means look for 3rd term	$u_n = n^{th}$ term This is a specific/particular term for any given n. For example: u_1 tells us what the 1st term is u_3 tells us what the 3rd term is Watch out: Sometimes courses use letters other than u (like a)	This is the sum of n terms. In other words, the sum up until a point. For example: S_2 tells us the sum of the first 2 terms S_3 tells us the sum of the first 3 terms

Let's demonstrate this with 2 examples:

If given a sequence: $\mathbf{3, 5, 7, 9, \ldots, 401}$	If given the n^{th} term: $\boldsymbol{u_n = 2n^2 - 4n + 1}$
All we know from this sequence is that: $u_1 = 3$ $u_2 = 5$ $u_3 = 7$ $u_4 = 9$ $u_n = 401$ (we don't know which term this sequence is so we call it u_n) We can also work out the sums of some terms $S_1 = 3$ $S_2 = 3+5 = 8$ $S_3 = 3+5+7 = 15$ $S_4 = 3+5+7+9 = 24$ We would like to be able to do things like: • Know what the 102nd term is straight away or any other high term in the sequence. Yes, we could write more terms down just by spotting the pattern of adding 2 each time and work this out, but this would be a bit tedious. • Know how many terms the sequence has, just by knowing that the last term is • Find the sum of a large number of terms • Find a rule to generate any term in the sequence (find any term we want, not just the ones that we are given). See the example on the right when we are given a rule.	We can calculate the terms here by plugging in values of n. The question always ready gives us the rule, so plugging values of n in generates the sequence for us. 1st term: Let $\boldsymbol{n = 1}$ $u_1 = 2n^2 - 4n + 1 = 2(1)^2 - 4(1) + 1 = -1$ 2nd term: Let $\boldsymbol{n = 2}$ $u_2 = 2n^2 - 4n + 1 = 2(2)^2 - 4(2) + 1 = 1$ 3rd term: Let $\boldsymbol{n = 3}$ $u_3 = 2n^2 - 4n + 1 = 2(3)^2 - 4(3) + 1 = 7$ 4th term: Let $\boldsymbol{n = 4}$ $u_4 = 2n^2 - 4n + 1 = 2(4)^2 - 4(4) + 1 = 17$ Our sequence looks like $\{-1, 1, 7, 17, \ldots\}$ Sum of the first 4 terms: $S_4 = -1 + 1 + 7 + 17 = 24$ It is great when we are given the rule. We can generate any term we want quickly. We are not always given the rule, so it is not as simple as always just being able to plug numbers in. We need a method to be able to generate the rule ourselves from a given sequence.

The limitations mentioned above mean we need a systematic way to deal with sequences. Fortunately, there are only 2 types of sequences that we have to deal (arithmetic and geometric). First of all, let's distinguish between these two types of sequences.

- **Arithmetic** sequences are where you **add** or **subtract** the SAME number which we call the **common difference**
- **Geometric** sequences are where you **multiply** or **divide** by the SAME number each time which we call the **common ratio**.

For these two sequences we use the following notations:

- a = first term. This is the same as saying u_1.
- d = common difference. This is the same number that you add or subtract by each time in an arithmetic sequence. We subtract any term by its previous term to get this.
- r = common ratio. This is the same number that you multiply or divide by each time in a geometric sequence. We divide any term by its previous term to get this.

Let's look at an example of each.

Consider the sequence $\mathbf{1, 3, 5, 7, 9, \ldots, 54}$	Consider the sequence $\mathbf{2, 4, 8, 16, \ldots, 512}$
We add 2 each time, so the series must be arithmetic • $d = 3 - 1 = 2$. Note: we could have also done $5 - 3 = 2$ or $= 7 - 5 = 2$ etc • $u_1 = a = 1$ (first term) • $u_n = 54$ (last term) • $S_3 = 1 + 3 + 5 = 9$	We multiply by 2 each time, so the series must be geometric • $r = \frac{4}{2} = 2$ Note: or we could have done $\frac{8}{4} = 2$ or $\frac{16}{8} = 2$ etc • $u_1 = a = 2$ (first term) • $u_n = 15$ (last term) • $S_3 = 2 + 4 + 8 = 14$

Method

Step 1: Ask yourself whether the series is arithmetic or geometric
Step 2: Ask yourself whether you know the first term is (a)
Step 3: Ask yourself whether you know what the common different d is (if arithmetic) or what the common ratio r is (if geometric)
Step 4: Plug into the formulae u_n and/or s_n (these formulae are given in the table on the page below)
Step 5: Solving using algebra knowledge (this could be an easy equation or a simultaneous equation)

Once you have understood all the above, it just a question of using either the formula or the definition.

- **Using the formulae** (your first thought should always be how can I use the formulae)

Arithmetic sequence	Geometric sequence
➢ $u_n = a + (n-1)d$ where d is the difference between any term and its previous term a is the first term ➢ $S_n = \frac{n}{2}[2a + (n-1)d]$ ➢ $S_n = \frac{n}{2}[a + l]$ where l is the last term Notice how there are two formulae for s_n. We use the second one when we know the last time.	➢ $u_n = ar^{n-1}$ where r is $\frac{\text{any term}}{\text{previous term}}$ a is the first term ➢ $S_n = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$ Note you can use either of these and I personally always use the first , but it makes life easier for students (in terms of avoiding negatives) if we use the first formula when r <1 and the second when r >1 ➢ $S_\infty = \frac{a}{1-r}$ This only can be used if the sequence converges (if $-1 < r < 1$)

How do we use the formulae? We either
- ✓ have the values of all the unknowns and can find what we want easily by plugging into the formula
- ✓ don't have the values of the knowns in which case we have to work backwards to solve for the unknown (harder questions will involve solving simultaneous equations)

Some examples:

Work out the nth term of 4,9,14,19, … i. Work out the 95th term	In an arithmetic sequence $u_1 = 2$, $u_{20} = 78$ and $u_n = 3710$ Find d and n	Consider the arithmetic sequence 3,9,15, … ,1353 Find the sum of the sequence	The 40th term of an arithmetic sequence is 106 and the sum of the first forty terms is 1900. Find the first term and the common difference
4 ,9, 14 ,19,… first term $= a = 4$ common difference $= d = 9 - 4 = 5$ We use the formula $u_n = a + (n-1)d$ Let's replace a and d in the formula $4 + (n-1)5$ Simplify $4 + 5n - 5$ $5n - 1$ i. Now that we have the rule we can work out any term 95th term means $n = 95$ $5(95) - 1 = 474$	We use the formula $u_n = a + (n-1)d$ We already know $a = u_1 = 2$ We also know $u_{20} = 78$. We use the formula for the lef- hand side: $2 + (20-1)d = 78$ $2 + (20-1)d = 78$ $76 = 19d$ $d = 4$ We also know $u_n = 3710$. We use the formula for the left-hand side: $2 + (n-1)(4) = 3710$ $2 + 4n - 4 = 3710$ $3712 = 4n$ $n = 928$	first term $= a = 3$ common difference $= d = 9 - 3 = 6$ We know the final term is 1353. We need to know what term number this corresponds to. We say $u_n = 1353$ We use the formula $u_n = a + (n-1)d$ $a + (n-1)d = 1353$ Now let's replace a and d $3 + (n-1)(6) = 1353$ $3 + 6n - 6 = 1353$ $6n = 1356$ $n = 226$ Now that we know we have 226 terms we can find the sum of the sequence using the formulas for s_n where $n = 226$ $S_{226} = \frac{226}{2}[2(3) + (226-1)6]$ $= 153228$	Here we don't know a and d, so we will need to work backwards to find them by using the formula The fortieth term of an arithmetic sequence is 106 tells us that $u_{40} = 106$ The sum of the first 40 terms of an arithmetic sequence tells us that $s_{40} = 1900$ Let's turn this into the u_n and s_n formulae $u_{40} = 106$: The formula tells us $a + 39d = 106$ $s_{40} = 1900$: The formula tells us $\frac{40}{2}[2a + 39(d)] = 1900$ So, we have two equations: $a + 39d = 106$ ① $20[2a + 39(d)] = 1900$ ② Simplifying both we get $a + 39d = 106$ $2a + 39d = 950$ Solve simultaneously $a = 12, d = -3$

The same method applies for geometric series, just with a different formula obviously.

- **Using the definition**:

Occasionally we are given the sequences in terms of unknowns and using the formula becomes problematic since there are too many unknowns.

- If a,b,c,d,… is an arithmetic sequence then we can build and solve the equation $b - a = c - b$
- If a,b,c,d,… is a geometric sequence then we can build and solve the equation $\frac{b}{a} = \frac{d}{c}$
- $u_n = s_n - s_{n-1}$ is useful if we are given the sum in terms of n and want to find u_n

The first three terms of an arithmetic sequence are $12 - p, 2p, 4p - 5$. Find the value of p and hence the first term and common difference of the series	The first three terms of a geometric sequence are $a, a + 14, 9a$. Find the value of a.
What do we know about an arithmetic sequence? If we subtract the successive terms, they must give the same answer. $2p - (12 - p) = (4p - 5) - 2p$ $2p - 12 + p = 4p - 5 - 2p$ $p = 7$ Put this back into sequence $12 - 7, 2(7), 4(7) - 5$ 5,14,23 $a = 5, d = 14 - 5 = 9$ Note: we could have also done $12 - p - 2p = 2p - (4p - 5)$ etc.	What do we know about and geometric sequence? If we divide the successive terms, they must give the same answer. $\frac{a}{a+14} = \frac{a+14}{9a}$ $9a^2 = (a + 14)^2$ $9a^2 = a^2 + 28a + 196$ $8a^2 - 28a - 196 = 0$ $2a^2 - 7a - 49 = 0$ $(2a + 7)(a - 7) = 0$ So, we actually have two solutions, $a = 7$ or $-\frac{7}{2}$.

Harder Types Of Questions

You will often see arithmetic and geometric series topics combined with other topics such as logs, trig, binomial expansion and sectors, so make sure your knowledge in these areas is also good.

Also, watch out for harder types of questions such as:

- **Greatest possible sum:** solve $u_n > 0$ to find out how many positive terms and then take the sum of these terms
- **Inequalities:** especially with logs. The sign swaps when multiplying or dividing by a negative. Remember log (number less than one) is negative
- **Solving geometric** series **simultaneously**: They are harder to solve, and the quickest method is to just divide successive terms in order to get unknowns to cancel
- Working out **how many years**:
  - ✓ if from beginning to year n to end of year m we do (m−n)+1 OR m−(n−1) i.e. we want to include n so don't take it away in the second formula.
  - ✓ if from beginning to year n to beginning of year m its $m - n$
- Questions with **lots of words**: We try and get these out of the words and into symbols u_n ands s_n as soon as possible and then follow our usual procedure of:
  Step 1: Ask yourself whether arithmetic or geometric
  Step 2: Ask yourself what a is
  Step 3: Ask yourself what d is (if arithmetic) or what r is (if geometric)
  Step 4: Plug into u_n and/or s_n

1 Bronze

1.1 Without Formulae

1) The nth term of a sequence is $2n - 3$
   i. Find the 42^{nd} term
   ii. Find the term number which is 1455

2) The nth term of a sequence is $(n + 3)(n - 4)$
   i. Find the 20^{th} term
   ii. Find the term number that has the value of 78

3) Find $\mathrm{u_1, u_2, u_3}$ and $\mathrm{u_{10}}$ of the sequence $\mathrm{u_n = 3n + 2}$

4) Find the value of n for which $\mathrm{u_n}$ has the given value, $\mathrm{u_n = 2n - 4, u_n = 24}$

5) The nth term of a sequence is $3{+}4n$.
   i. Find the first 3 terms of the sequence.
   ii. Find the value of n for which $\mathrm{u_n}$ has the value 27
   iii. Find S_5

6) Find $\mathrm{S_5}$ where $\mathrm{u_n = 2n + 5}$

7) The first three terms of an arithmetic sequence are $12 - p, 2p, 4p - 5$ respectively, where p is a constant. Find the value of p and hence the first term and common difference of the series

8) The first three terms of a geometric sequence are $a, a + 14, 9a$. Find the value of a

9) Consider the geometric sequence $x - 3$, $x + 1$, $2x + 8$. When $x = 5$, the series is geometric
   i. Write down the first three terms
   ii. Find the common ratio
   iii. Find the other value of x for which the sequence is geometric
   iv. For this value of x, find the common ratio

1.2 Using Formulae

10) Work out the nth term of $4, 9, 14, 19, \ldots$

11) The first three terms of an arithmetic sequence are $36, 40, 44$
    i. Find the common difference
    ii. Find $\mathrm{u_n}$
    iii. Show that $\mathrm{s_n = 2n^2 + 34n}$
    iv. Hence write down the value of $\mathrm{s_{14}}$

12) Consider the arithmetic sequence $2,5,8,11,\ldots$
    i. Find u_{101}
    ii. Find the value of n so that $u_n = 152$

13) An arithmetic sequence is given by $5,8,11,\ldots$
    i. Write down the value of d
    ii. Find u_{100}
    iii. Find s_{100}
    iv. Given that $u_n = 1502$, find the value of n

14) Let s_n, be the sum of the first n terms of the arithmetic series $2+4+6+\cdots$
    i. Find s_4
    ii. Find s_{100}

15) In an arithmetic sequence $u_1 = 2$ and $u_2 = 8$
    i. Find d
    ii. Find u_{20}
    iii. Find s_{20}

16) The first term of a geometric progression is 12 and the second term is -6. Find
    i. The tenth term
    ii. The sum to infinity

17) In an arithmetic sequence $u_1 = 2$, $u_{20} = 78$ and $u_n = 3710$
    i. Find d
    ii. Find the value of n

18) In an arithmetic sequence the first term is 5 and the fourth term is 40, find the second term

19) In an arithmetic sequence the first term is -7 and the sum of the first 20 terms is 620
    i. Find the common difference
    ii. Find the value of the $78th$ term

2 Silver

2.1 Without Formulae

20) The sum of the first n terms of a series is given by is $s_n = 2n^2 - n$, where $n \in \mathbb{Z}^+$ Find the first three terms of the series and find an expression for the n^{th} term of the sequence, giving your answer in terms of n.

21) The sum s_n of the n terms of an arithmetic progression is given by $s_n = 32n - n^2$. Find the first term and common difference.

22) The sum of the first n terms of an arithmetic sequence is given by $s_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m, and u_{32}, are consecutive terms of a geometric sequence. Find m.

2.2 Using Formulae

2.2.1 Finding n

23) Consider the geometric sequence $5, 10, \ldots, 1280$. Find the number of terms in the sequence.

24) Consider the arithmetic sequence $3, 9, 15, \ldots, 1353$. Find the sum of the sequence

25) Find the sum of all the multiples of 3 between 100 and 1000

2.2.2 Simultaneous Equations

26) The fourth term of an arithmetic sequence is 3 and the sum of the first 6 terms is 27. Find the first term and the common difference

27) The 40th term of an arithmetic sequence is 106 and the sum of the first forty terms is 1900. Find the first term and the common difference

28) The fourth term of a geometric is 10 and the seventh term of the series is 80. For this series, find
    i. The common ratio
    ii. The first term
    iii. The sum of the first 20 terms, giving your answer to the nearest whole number

29) The sum to infinity of a geometric sequence is 16 and the sum of the first four terms is 15. Find the possible first terms and common ratios.

30) The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8. Find the first term and common difference.

31) The sum of the first three terms of a geometric sequence is 62.755, and the sum of the infinite sequence is 440. Find the common ratio.

32) The sum of the first 2 terms of a geometric sequence is 6 and the sum of the first 4 terms of a geometric sequence is 30. Find the first term and common ratio

2.2.3 With Inequalities

33) Find the **greatest possible sum** of the arithmetic sequence $85, 78, 71$

34) The third term of an arithmetic sequence is 1407 and the $10th$ term is 1183.
    i. Find the first term and common difference of the sequence
    ii. Calculate the number of **positive terms** of the sequence

35) An arithmetic sequence has first term 1000 and common difference of -6. The sum of the first n terms of this sequence is **negative**. Find the **least value** of n

36) The 5th term of an arithmetic series is 16 and the 10th term is 30
    i. Find the first term and common difference
    ii. How many terms of the series are needed for the sum of the series to **exceed** 1000?

37) Find the least number of terms required for the sum of $4 + 9 + 14 + 19 + \cdots$ to **exceed** 2000.

2.2.4 Worded Questions

38) Carol starts a new job on a salary of $£20{,}000$. She is given an annual wage rise of $£500$ at the end of every year until she reaches her maximum salary of $£25{,}000$. Find the total she earns (assuming no other rises)
    i. In the first 10 years
    ii. Over 15 years
    iii. State one reason why this may be an unsuitable model

39) A company extracted 4500 tonnes of minerals from a mine during 2018. The mass of the mineral which the company expects to extract in each subsequent year is modelled to decrease by 2% each year.
    i. Find the total mass of the mineral which the company expects to extract from 2018 to 2040 inclusive, giving your answer to 3 significant figures
    ii. Find the mass of the mineral which the company expects to extract during 2040, giving your answer to 3 significant figures

    The costs of extracting the mineral each year are assumed to be
    $£800$ per tonne for the first 1500 tones
    $£600$ per tonne for any amount in excess of 1500 tonnes
    The expected cost of extracting the mineral from 2018 to 2040 inclusive is $£x$ million
    iii. Find the value of x

40) A small company which makes batteries for electric cars has a 10-year plan for growth
- In year 1 the company will make $2{,}600$ batteries
- In year 10 the company aims to make $12{,}000$ batteries

In order to calculate the number of batteries it will need to make each year, form year 2 to year 9, the company considers two models, Model A and Model B

In Model A the number of batteries male will increase by the same **number** each year
    i. Using Model A, determine the number of batteries the company will make in year 2

In model B the number of batteries will increase by the same **percentage** each year
    ii. Using model B, determine the number of batteries the company will make in year 2. Give you answer to the nearest 10 batteries.

Sam calculates the total number of batteries make from year 1 to year 10 inclusive using each of the 2 models
    iii. Calculate the difference between the two totals, giving your answer to the nearest 100 batteries

41) Lewis played a game of space invaders. He scored points for each spaceship that he captured. Lewis scored 140 points for capturing his first spaceship. He scored 160 points for capturing his second spaceship, 180 points for capturing his third spaceship, and so on. The number of points scored for capturing each successive spaceship formed an arithmetic sequence.
   i. Find the number of points that Lewis scored for capturing his $20th$ spaceship
   ii. Find the total number of points Lewis scored for capturing his first 20 spaceships

   Sian played an adventure game. She scored points for each dragon that she captured. The number of points that Sian scored for capturing each successive dragon formed an arithmetic sequence. Sian captured n dragons and the total number of points that she scored for capturing all n dragons was 8500. Given that Sian scored 300 points for capturing her first dragon and then 700 points for capturing her n^{th} dragon,
   iii. find the value of n

2.2.4.1 *Being Careful with difference between years*

42) **Finding n**: Jess started work 20 years ago. In year 1 her annual salary was $£17,000$. Her annual salary increased by $£1,500$ each year, so that her annual salary in year 2 was $£18,500$, in year 3 it was $£20,000$ and so on, forming an arithmetic sequence. This continued until she reached her maximum annual salary of $£32,000$ in year k. Her annual salary then remained at $£32,000$.
   i. Find the value of the constant k
   ii. Calculate the total amount that Jess has earned in the 20 years

43) On John's 10th birthday he received the first of an annual birthday gift of money from his uncle. This first gift was $£60$ and on each subsequent birthday the gift was $£15$ more than the year before. The amounts of these gifts form an arithmetic sequence
   i. Show that, after his 12th birthday, the total of these gifts was $£225$
   ii. Find the amount that John received from his uncle as a birthday gift on his 18th birthday.
   iii. Find the total of these birthday gifts that John had received from his uncle up to and including his 21st birthday.

   When John had received n of these birthday gifts, the total money that he had received from these gifts was $£3375$
   iv. Show that $n^2 + 7n = 25 \times 18$
   v. Find the value of n, when he had received $£3375$ in total, and so determine John's age at this time

44) A company, which is making 200 mobile phones each week, plans to increase its production. The number of mobile phones produced is to be increased by 20 each week from 200 in week 1 to 220 in week 2, to 240 in week 3 and so on, until it is producing 600 in week N
   i. Find the value of N

   The company then plans to continue to make 600 mobile phones each week
   ii. Find the total number of mobile phones that will be made in the first 52 weeks starting from and including week 1

45) A company, which is making 140 bicycles each week, plans to increase its production. The number of bicycles produced is to be increased by d each week from 140 in week 1 to $140 + d$ in week 2, to $140 + 2d$ in week 3 and so on, until it is producing 206 in week 12.
   i. Find the value of d

   After week 12 the company plans to continue to make 206 bicycles each week
   ii. Find the total number of bicycles that would be made in the first 52 weeks starting from and including week 1

46) Portable telephones are first sold in the country Cellmania in 1990. During 1990, the number of units sold is 160. In 1991, the number of units sold is 240 and in 1992, the number of units sold is 360. In 1993 it was noticed that the annual sales formed a geometric sequence with first term 160, the $2nd$ and $3rd$ terms being 240 and 360 respectively
   i. How many units are sold during 2002?
   ii. In what year does the number of units sold first exceed 5000?
   iii. Between 1990 and 1992 the total number of units sold is 760. What is the total number of units sold between 1990 and 2002?

3 Gold

3.1 Without Formulae

47) The first three terms of a geometric sequence are $\frac{k-3}{2}, \frac{2k+3}{4}$ *and* $\frac{12k+3}{8}$
   i. Show that k satisfies the equation $8k^2 - 45k - 18 = 0$
   ii. Find the first 5 terms of the sequence
   Hint: Write out the sequence after finding k and the term and common ratio

48) Find two distinct numbers p and q such that $p, q, 10$ are in arithmetic progression and $q, p, 10$ are in geometric progression.

3.2 Using Formulae

3.2.1 Simultaneous Equations

49) In a geometric series, the sum of the second and third term is -12 and the sum of the third and fourth term is -36. Find the common ratio.

50) The sum of the first two terms of a geometric progression is 8 and the sum of the next two terms is 2. Find the possible values of the common ratio and for each one. Find the first term of the progression.

51) The sum of the 1st and 2nd terms of a geometric progression is 50 and the sum of the 2nd and 3rd terms is 30. Find the sum to infinity.

52) In a geometric sequence, the first term is $3a$ and the common ratio is r. In another geometric sequence the first term is a and the common ratio is $-2r$. The sums to infinites are equal for both sequences. Find the common ratio.

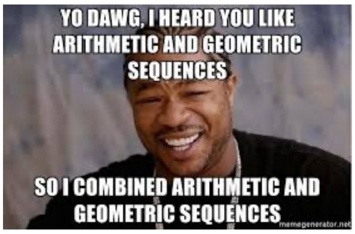
53) A geometric series is such that s_{10} is four times S_5. Find the exact value of r.

54) In a geometric sequence, the fourth term is 8 times the first term. The sum of the first 10 terms is 2557.5. Find the 10th term of this sequence.

55) The first term of an infinite geometric series exceeds the second term by 9. The sum of the series is 81. Find the common ratio.

56) An arithmetic progression is such that the eighth term is three times the third term. Show that the sum of the first eight terms is four times the sum of the first four terms.

3.2.2 Arithmetic and Geometric Together In One Question



57) The 1st, 2nd and 3rd terms of a geometric progression are the 1st, 9th and 21st terms respectively of an arithmetic progression. The 1st term of each progression is 8 and the common ratio of the geometric is r, where $r \neq 1$. Find
  i. The value of r
  ii. The 4th term of each sequence

58) The 1st term of an arithmetic progression is 2. The 1st, 2nd and 5th terms of this progression are the first 3 terms of a geometric progression respectively. Find
  i. The common difference of the arithmetic progression
  ii. The common ratio of the geometric progression

59) A geometric sequence $u_1, u_2, u_3, \ldots$ has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$
  i. Find the common ratio of the geometric sequence.

  An arithmetic sequence $v_1, v_2, v_3, \ldots$ is such that $v_2 = u_2$ and $v_4 = u_4$
  ii. Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$

60) An arithmetic sequence $\{u_n : n \in \mathbb{Z}^+\}$ has first term $u_1 = 1.6$ and a common difference $d = 1.5$. The geometric sequence $\{v_n : n \in \mathbb{Z}^+\}$ has first term $v_1 = 3$ and common ratio $r = 1.2$
  i. Find an expression for $u_n - v_n$ in terms of n
  ii. Determine the set of values of n for which $u_n > v_n$
  iii. Determine the greatest value of $u_n - v_n$. Giving your answer correct to four significant figures

61) The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence. The arithmetic sequence has first term a and non-zero common difference d.
  i. Show that $d = \frac{a}{2}$
  ii. The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200. Find the least value of n for which this occurs.

62) A sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by formula $s_n = 4n^2 - 2n$. Three terms of sequence u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m

3.2.3 Worded Questions

63) Xin has been given a 14-day training schedule by her coach. Xin will run for A minutes on day 1, where A is a constant. She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.
  i. Show that on day 14, Xin will run for $(A + 13d + 13)$ minutes.

  Yi has also been given a 14-day training schedule by her coach. Yi will run for $(A - 13)$ minutes on day 1. She will then increase her running time by $(2d - 1)$ minutes each day. Given that Yi and Xin will run for the same length of time on day 14,
  ii. find the value of d.

  Given that Xin runs for a total time of 784 minutes over the 14 days,
  iii. find the value of A

64) A store begins to stock a new range of DVD players and achieves sales of £1500 of these products during the first month. In a model it is assumed that sales will decrease by £x in each subsequent month, so that sales of £$(1500 - x)$ and £$(1500 - 2x)$ will be achieved in the second and third months respectively. Given that sales total £8100 during the first six months, use the model to
  i. find the value of x
  ii. find the expected value of sales in the eighth month
  iii. show that the expected total of sales in pounds during the first n months is given by $kn(51 - n)$, where k is an integer to be found.
  iv. Explain why this model cannot be valid over a long period of time

65) A farmer has a pay scheme to keep fruit pickers working throughout the 30 day season. He pays $£a$ for their first day, $£(a + d)$ for their second day, $£(a + 2d)$ for their third day, and so on, thus increasing the daily payment by £d for each extra day they work. A picker who works for all 30 days will earn $£40.75$ on the final day.

i. Use this information to form an equation in a and d.

A picker who works for all 30 days will earn a total of $£1005$

ii. Show that $15(a + 40.75) = 1005$

iii. Hence find the value of a and the value of d

66) Shelim starts his new job on a salary of $£14\,000$. He will receive a rise of $£1500$ a year for each full year that he works, so that he will have a salary of $£15\,500$ in year 2, a salary of $£17\,000$ in year 3 and so on. When Shelim's salary reaches $£26\,000$, he will receive no more rises. His salary will remain at $£26\,000$.

i. Show that Shelim will have a salary of $£26\,000$ in year 9

ii. Find the total amount that Shelim will earn in his job in the first 9 years

Anna starts her new job at the same time as Shelim on a salary of $£A$. She receives a rise of $£1000$ a year for each full year that she works, so that she has a salary of $£(A + 1000)$ in year 2, $£(A + 2000)$ in year 3 and so on. The maximum salary for her job, which is reached in year 10, is also $£26\,000$.

iii. Find the difference in the total amount earned by Shelim and Anna in the first 10 years

3.2.4 With Logs

67) An arithmetic sequence has first term $\ln a$ and common difference $\ln 3$. The 13th term of the sequence is $8ln9$. Find the value of a.

68) The first two terms of an infinite geometric sequence in order are $2\,log_2 x, log_2 x$ where $x > 0$.

i. Find r

ii. Show that the sum of the infinite sequence is $4\log_2 x$

The first three terms of an arithmetic sequence in order are

$$log_2 x, log_2 \frac{x}{2}, log_2 \frac{x}{4} \text{ where } x > 0$$

iii. Find d, giving your answer as an integer

Let S_{12} be the sum of the first 12 terms of the arithmetic sequence

iv. Show that $S_{12} = 12\log_2 x - 66$

v. Given that S_{12} is equal to half the sum of the infinite geometric sequence, find x, giving your answer in the form 2^p, where $p \in \mathbb{Q}$

3.2.5 With Inequalities and Logs

69) How many terms are needed for the sum of the geometric series $3 + 6 + 12 + 24 + \cdots$ to **exceed** $100{,}000$

70) The second and third terms of a geometric series are 192 and 144 respectively. Find the smallest value of n for which the sum of the first n terms of the series exceeds 1000

71) The first term of a geometric series is 120. The sum to infinity of the series is 480. The sum of the n terms of the series is greater than 300. Calculate the smallest possible values of n.

72) In a geometric series $u_1 = \frac{1}{81}$ and $u_4 = \frac{1}{3}$

i. Find the value of r

ii. Find the **smallest value** of n which $s_n > 40$

73) The first three term of a geometric sequence are $u_1 = 0.64, u_2 = 1.6$, and $u_3 = 4$.

i. Find the value of r

ii. Find the value of s_6

iii. Find the **least value** of n such that $s_n > 75000$

74) Carlos saves money every year. The first year he saves $£100$. Each year he increases the amount he saves by 10%. After how many years do Carlos's savings first exceed $£1000$ (excluding any interest he has earned)

75) A geometric series has first term 5 and common ratio $\frac{4}{5}$. Given that the sum of the k terms of the series is greater than 24.95

i. Show that $k > \frac{\log 0.002}{\log 0.8}$

ii. Find the smallest possible value of k

76) The first term of a geometric series is 20 and the common ratio is $\frac{7}{8}$. The sum of the N terms of the series is S_N. Find the smallest value of N, for which $S_\infty - S_N < 0.5$

77) The adult population of a town is $25{,}000$ at the end of year 1. A model predicts that the adult population of the town will increase by 3% each year, forming a geometric sequence

i. Show that the predicted adult population at the end of year 2 is $25{,}750$

ii. Write down the common ratio of the geometric sequence

The model predicts that year N will be the first year in which the adult population of the town exceeds $40{,}000$

iii. Show that $(N-1)\log 1.03 > \log 1.6$

iv. Find the value of N

At the end of each year, each member of the adult population of the town will give £1 to a charity fund. Assuming the population model,

v. Find the total amount that will be given to the charity fund for the 10 years from the end of year 1 to the end of year 10, giving your answer to the nearest £1000.

3.2.6 With Trig

78) Find the set of values of θ, such that $-\pi < \theta < \pi$, for which the geometric progression $1 + 2\cos^2\theta + 4\cos^4\theta + 8\cos^6\theta + \cdots$ has a sum to infinity.

i. Show that for this value of θ, the sum to infinity of the progression is $-\sec 2\theta$.

79) Given that the first three terms of a geometric series are

$$12\cos\theta \qquad 5 + 2\sin\theta \qquad \text{and} \qquad 6\tan\theta$$

i. Show that

$$4\sin^2\theta - 52\sin\theta + 25 = 0$$

Given that θ is an obtuse angle measures in radians,

ii. Solve the equation in part i. to find the exact value of θ

iii. Show that the sum to infinity of the series can be expressed in the form

$$k(1-\sqrt{3})$$

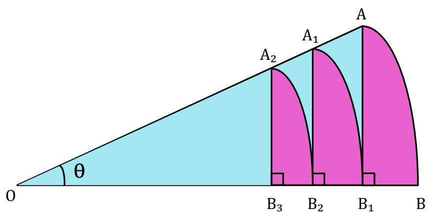
where k is a constant to be found

3.2.7 With Sectors

80) A circular plank is cut into 12 sectors whose areas are in arithmetic progression. If the area of the largest sector is twice that of the smallest, find the angle in terms of π between the straight edges of the smallest sector.

81) A circular disc is cut into 12 sectors whose areas are in arithmetic progression. If the angle of the largest sector is twice that of the smallest sector, find the size of the angle of the smallest sector

82) The diagram shows a sector AOB of a circle of radius 1 with centre O, where angle $AOB = \theta$. The lines (AB_1), (A_1B_2), (A_2B_3) are perpendicular to OB. (A_1B_1), (A_2B_2) are all arcs of circles with centre O



Calculate the sum to infinity of the arc lengths $AB + A_1B_1 + A_2B_2 + A_3B_3 + \cdots$

4 Diamond

4.1 Using Formulae

4.1.1 Simultaneous Equations

83) An arithmetic sequence has first term 15 and second term 19. Another arithmetic sequence has first term 420 and second term 415. The sum of each sequence is the same for a particular n. What is n?

84) Find in terms of k, the 50^{th} term of the arithmetic sequence $(2k+1),(4k+4),(6k+7)$. ... Give your answer in its simplest form

85) A geometric series has first term b^2-13, common ratio $\frac{1}{\mathrm{b}}$ and sum to infinity -6. Find all possible values of the common ratio.

86) In the arithmetic series $k+2k+3k+\cdots+100$, k is a positive integer and k is a factor of 100.
   i. Find, in terms of k, an expression for the number of terms in the series
   ii. Show that the sum of the series is $50+\frac{5000}{\mathrm{k}}$

87) The first term in an arithmetic series is $5t+3$, where t is a positive integer. The last term is $17t+11$ and the common difference is 4. Show that the sum of the series is divisible by 12 when, and only when, t is odd.

4.1.2 Arithmetic and Geometric Together in One Question

88) The sums of the terms of a sequence follow the pattern $S_1=1+k$, $\mathrm{S}_2=5+3\mathrm{k}, \mathrm{S}_3=12+7\mathrm{k}, \mathrm{S}_4=22+15\mathrm{k}, \ldots$, where $\mathrm{k} \in \mathbb{Z}$.
   i. Given that $\mathrm{u}_1=1+\mathrm{k}$, find $\mathrm{u}_2, \mathrm{u}_3$, and u_4 $(ans\ 4+2k, 7+4k, 10+8k)$
   ii. Find a general expression for u_n $(ans=3n-2+2^{n-1}k)$

89) The first 3 terms of a geometric series are also the 3rd, 14th and 58th terms of an arithmetic series. Given the sum of the first 3 terms of the arithmetic series is 24, and the sum of the first 5 terms is 55, find the sum of the first 5 terms of the geometric sequence

90) The sum of the first three numbers in an arithmetic sequence is 24. If the first number is decreased by 1 and the second number is decreased by 2, then the third number and the two new numbers are in geometric sequence. Find all possible sets of three numbers which are in the arithmetic sequence.

91) The n^{th} term of a geometric progression is denoted by g_n and the n^{th} term of an arithmetic progression is denoted by a_n. It is given that $\mathrm{a}_1=\mathrm{g}_1=1+\sqrt{5}$, $\mathrm{g}_3=\mathrm{a}_2$ and $\mathrm{g}_4+\mathrm{a}_3$ =0. Given also that the geometric progression is convergent, show that it's sum to infinity is $4+2\sqrt{5}$.

92) Each of the terms of an arithmetic series is added to the corresponding term of a geometric series, forming a new series with first term $\frac{3}{8}$ and second term $\frac{13}{16}$. The common difference of the arithmetic series is four times

as large as the first term of the geometric series. The common ratio of the geometric series is twice as large as the first term of the arithmetic series. Determine the possible value of the first term of the geometric series.

4.2 With Binomial Expansion

93) Given that the coefficients of x, x^2 and x^4 in the expansion of $(1+kx)^n$, where $n \geq 4$ and k is a positive constant, are the consecutive terms of a geometric sequence, show that $k = \frac{6(n-1)}{(n-2)(n-3)}$.

4.3 With Logs

94) The first terms of an arithmetic sequence are $\frac{1}{log_2 x}, \frac{1}{log_8 x}, \frac{1}{log_{32} x}, \frac{1}{log_{128} x}, \ldots$. Find x if the sum of the first 20 terms of the sequence is equal to 100
Hint: d simplified is $\frac{2}{log_2 x}$, and $a = \frac{1}{log_2 x}$. Plug into s_n

Sequences & Series: Arithmetic & Geometric Solutions

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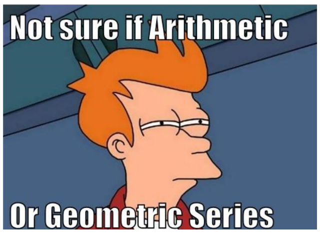

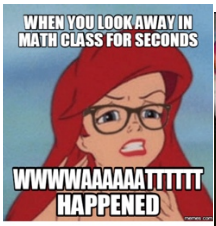

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1 Bronze

1.1 Without Formulae

1)

$$2n - 3$$

i.

n represents the term number

We want the 42nd term and hence we replace n with 42

$$2(42) - 3 = 81$$

ii. This is telling us that we want n (the term number) which gives a result of 1455

$$2n - 3 = 1455$$

We solve for n

$$2n - 3 = 1455$$
$$2n = 1458$$
$$n = 729$$

729^{th} term

2)

$$(n + 3)(n - 4)$$

i.

n represents the term number

We want the 20th term and hence we replace n with 20

$$(20 + 3)(20 - 4) = 368$$

ii. This is telling us that we want n (the term number) which gives a result of 78

$$(n + 3)(n - 4) = 78$$

We solve for n

$$(n + 3)(n - 4) = 78$$

Let's expand the brackets

$$n^2 - n - 12 = 78$$

We have a quadratic which means we need to get zero on one side and factorise or use the quadratic formula (this worksheet assumes you know how to solve quadratics and therefore does not go into detail on this).

$$n^2 - n - 90 = 0$$
$$(n + 9)(n - 10) = 0$$
$$n = -9 \text{ or } 10$$

But clearly n has to be a positive integer (since it represents term number), so $n = 10$.

10^{th} term

3)

We have $u_n = 3n + 2$. To find u_n, we simply plug in the required values of n.
We plug in $n = 1, 2, 3, 10$.

$$u_1 = 3(1) + 2 = 5$$
$$u_2 = 3(2) + 2 = 8$$
$$u_3 = 3(3) + 2 = 11$$
$$u_{10} = 3(10) + 2 = 32$$

4)

We are given that $u_n = 24$, and $u_n = 2n - 4$. So we can form an equation by setting them equal

$$24 = 2n - 4$$

Now we solve for n

$$28 = 2n$$
$$n = 14$$

5)

i.

We are given the n^{th} term as $u_n = 3 + 4n$
In order to find the terms asked for we replace every n we see with the values of the term number

first term $= u_1 = 3 + 4(1) = 7$
second term $= u_2 = 3 + 4(2) = 11$
third term $= u_3 = 3 + 4(3) = 15$

ii.

$u_n = 24$
We are given that $u_n = 3 + 4n$
So, we can form an equation

$$3 + 4n = 27$$
$$4n = 24$$
$$n = 6$$

iii.

Our sequence from part i. looks like $7, 11, 15, \dots$
We can see that we add 4 each time so we can find more of the sequence

$$7, 11, 15, 19, 23, 27, \dots$$

$S_5 = \text{sum of the first 5 terms} = 7 + 11 + 15 + 19 + 23 = 75$

6)

Remember, S_5 is the sum of first 5 terms. $S_5 = u_1 + u_2 + u_3 + u_4 + u_5$.

There are multiple ways of solving this problem

Method 1.
We use the formula $S_n = \frac{n}{2}[a + l]$, where a is the first term, l is the last ter

We work out the first term $a = u_1 = 2(1) + 5 = 7$
We work out the last term $l = u_5 = 2(5) + 5 = 15$

$$S_5 = \frac{5}{2}(7 + 15) = 55$$

Method 2.
We use the formula $S_n = \frac{n}{2}[2a + (n-1)d]$, where a is the first term, d is the common difference.

We work out the first term $a = u_1 = 2(1) + 5 = 7$.
Common difference $d = u_2 - u_1 = 2(2) + 5 - 7 = 2$.
In fact, d is always the coefficient of n in an arithmetic sequence, can you see why?

$$S_5 = \frac{5}{2}(2(7) + (5 - 1)2) = 55$$

Method 3.

Work out u_1 to u_5 individually and add them up (not a good method when we have a number much bigger than 5).

$$u_1 = 2(1) + 5 = 7$$
$$u_2 = 2(2) + 5 = 9$$
$$u_3 = 2(3) + 5 = 11$$
$$u_4 = 2(4) + 5 = 13$$
$$u_5 = 2(5) + 5 = 15$$

$$S_5 = 7 + 9 + 11 + 13 + 15 = 55$$

7)

What do we know about an arithmetic sequence? If we subtract the successive terms, they must give the same answer.

$$2p - (12 - p) = (4p - 5) - 2p$$
$$2p - 12 + p = 4p - 5 - 2p$$
$$p = 7$$

Put this back into sequence

$$12 - 7, 2(7), 4(7) - 5$$
$$5,14,23$$
$$a = 5, d = 14 - 5 = 9$$

Note: we could have also done $12 - p - 2p = 2p - (4p - 5)$ etc.

8)

We approach it in a similar manner to the previous question. We know the quotient of successive terms are equal.

$$\frac{a}{a+14} = \frac{a+14}{9a}$$

$$9a^2 = (a+14)^2$$

$$9a^2 = a^2 + 28a + 196$$

$$8a^2 - 28a - 196 = 0$$

$$2a^2 - 7a - 49 = 0$$

$$(2a + 7)(a - 7) = 0$$

So, we actually have two solutions, $a = 7 \text{ or } -\frac{7}{2}$.

9)

i. Plug in $x = 5$ to $x - 3, x + 1, 2x + 8$:

$$5 - 3, 5 + 1, 2(5) + 8$$
$$2, 6, 18$$

ii.

$$r = \frac{u_2}{u_1} = \frac{6}{2} = 3$$

iii. If the sequence is geometric, then the ratio of successive terms must be the same, so we can build the following equation:

$$\frac{x-3}{x+1} = \frac{x+1}{2x+8}$$

Now we solve by cross multiplying

$$(2x+8)(x-3)=(x+1)^2$$

Expand the brackets

$$2x^2+2x-24=x^2+2x+1$$
$$x^2-25=0$$
$$x=\pm 5$$

The other value of x for which the sequence is geometric is -5.

iv. We first plug in $x=-5$ into $x-3, x+1$ to find the first two terms.

$$(-5)-3,(-5)+1$$
$$-8,-4$$
$$r=\frac{u_2}{u_1}=\frac{-4}{-8}=\frac{1}{2}$$

1.2 Using Formulae

10)

4 ,9, 14 ,19,…

first term $= a = 4$

common difference $= d = 9-4=5$

The formula for $u_n = a+(n-1)d$

Let's replace a and d in the formula

$$4+(n-1)5$$
$$4+5n-5$$
$$5n-1$$

11)

i.

The common difference $d=u_2-u_1=40-36=4$

ii.

We use the formula $u_n = a+(n-1)d$. a is the first term, which is 36.

$$u_n=36+(n-1)4$$
$$=36+4n-4$$
$$=32+4n$$

iii.

We use the formula $S_n=\frac{n}{2}[2a+(n-1)d]$. We already have a, d from the previous part.

$$S_n=\frac{n}{2}(2(36)+(n-1)(4))$$
$$=\frac{n}{2}(72+4n-4)$$
$$=\frac{n}{2}(68+4n)$$
$$=n(34+2n)$$
$$=2n^2+34n$$

iv.

$$s_{14}=2(14)^2+34(14)=868$$

12)

i.

Let's work out the n^{th} term so we can substitute 101 in.

$$u_n=a+(n-1)d$$

The first term $= a = 2$

The common difference $= d = 5-2=3$

$$u_n=2+(n-1)(3)$$
$$=2+3n-3$$
$$=3n-1$$

Now, $u_{101}=3(101)-1=302$.

ii.

We have the equation $152=u_n=3n-1$, we solve for n.

$$152=3n-1$$

$$153 = 3n$$
$$n = 51$$

13)

i.

d is the common difference, $d = 8 - 5 = 3$

ii.

We first need to work out the n^{th} term.

$$u_n = a + (n-1)d$$

We already know $d = 3$, a is the first term, which is 5.

$$u_n = 5 + (n-1)(3)$$
$$u_n = 5 + 3n - 3$$
$$u_n = 2 + 3n$$

Now, plug in $n = 100$, $u_{100} = 2 + 3(100) = 302$.

iii.

We use the formula $S_n = \frac{n}{2}[a + l]$, where l is the last term. The reason we use this formula is because we already know the last term $l = u_{100} = 302$.

So, $s_{100} = \frac{100}{2}(5 + 302) = 15350$.

14)

i.

The common difference $d = 4 - 2 = 2$. So, the next term is $6 + 2 = 8$.

$$s_4 = u_1 + u_2 + u_3 + u_4$$
$$s_4 = 2 + 4 + 6 + 8 = 20$$

ii.

The method above is obviously not practical for finding s_{100}. We use the formula $s_n = \frac{n}{2}[2a + (n-1)d]$ instead.

We have the first term $a = 2$.
The common difference $d = 2$.
We also have $n = 100$.

$$s_{100} = \frac{100}{2}\left(2(2) + (100-1)(2)\right)$$
$$= 50(4 + 2(100) - 2)$$
$$= 50(202)$$
$$= 10100$$

15)

i.

$$d = u_2 - u_1 = 8 - 2 = 6$$

ii.

Using the formula $u_n = a + (n-1)d$
We know a is the first term, which is 2.

$$u_{20} = 2 + (20-1)(6)$$
$$= 2 + 114 = 116$$

iii.

We use the formula $S_n = \frac{n}{2}[a + l]$, we already have $a = 2$, and we worked out $l = u_{20} = 116$ from the previous part.

$$s_{20} = \frac{20}{2}(2 + 116) = 1180$$

16)

i.

We can use the formula $u_n = ar^{n-1}$.
We know the first term $a = 12$.
We can work out the common ratio $r = \frac{-6}{12} = -\frac{1}{2}$.

We find the tenth term $= u_{10} = 12\left(-\frac{1}{2}\right)^{10-1} = \frac{12}{-2^9} = -\frac{3}{2^7} = -\frac{3}{128}$

ii.

We use the formula $S_\infty = \frac{a}{1-r}$. We need to check that $-1 < r = -\frac{1}{2} < 1$, before we can use this formula, which is indeed the case.

$$S_\infty = \frac{12}{1-\left(-\frac{1}{2}\right)} = \frac{12}{\frac{3}{2}} = 8$$

17)

i.

We can use the formula $u_n = a + (n-1)d$ to find d, since we already know $a = u_1 = 2$.
If we plug in $n = 1$, then d would disappear, so we need to use $n = 20$.

$$u_{20} = 78 = 2 + (20-1)d$$
$$76 = 19d$$
$$d = 4$$

ii.

We now solve for n using the same formula.

$$u_n = 3710 = 2 + (n-1)(4)$$
$$3710 = 2 + 4n - 4$$
$$3712 = 4n$$
$$n = 928$$

18) In an arithmetic sequence the first term is 5 and the fourth term is 40, find the second term

We can use the formula $u_n = a + (n-1)d$.

We already have $a = u_1 = 5$.
We can solve for d using our knowledge of u_4.

$$u_4 = 40 = 5 + (4-1)d$$
$$35 = 3d$$
$$d = 5$$

Now, we plug these values back in.

$$u_2 = 5 + (2-1)(5) = 10$$

19)

i. We can use the formula $s_n = \frac{n}{2}[2a + (n-1)d]$.
We already know the first term $a = -7$.

$$s_{20} = 620 = \frac{20}{2}(2(-7) + (20-1)d)$$
$$620 = 10(-14 + 19d)$$
$$620 = -140 + 190d$$
$$760 = 190d$$
$$d = 4$$

The common difference is 4.

ii. We use the formula $u_n = a + (n-1)d$

$$u_{78} = -7 + (78-1)(4) = 301$$

2 Silver

2.1 Without Formulae

20)

Way 1:
Let's substitute $n = 1, 2, 3$ into s_n to find the first three terms of the series.

$$s_1 = 2(1)^2 - 1 = 1$$
$$s_2 = 2(2)^2 - 2 = 6$$
$$s_3 = 2(3)^2 - 3 = 15$$

We can find the corresponding u_n using the formula $u_n = s_n - s_{n-1}$, and that $u_1 = s_1$:

$$u_1 = s_1 = 1$$
$$u_2 = s_2 - s_1 = 6 - 1 = 5$$
$$u_3 = s_3 - s_2 = 15 - 6 = 9$$

To find the n^{th} term of the sequence, we need to find a, d.

$$a = u_1 = 1$$
$$d = u_2 - u_1 = 5 - 1 = 4$$

$$u_n = a + (n-1)d$$
$$u_n = 1 + (n-1)4$$
$$u_n = 1 + 4n - 4$$
$$u_n = 4n - 3$$

Way 2:

$$u_n = s_n - s_{n-1}$$
$$= 2n^2 - n - \left(2(n-1)^2 - (n-1)\right)$$
$$= 2n^2 - n - [2(n-1)(n-1) - n + 1]$$
$$= 2n^2 - n - [2(n^2 - 2n + 1) - n + 1]$$
$$= 2n^2 - n - [2(n^2 - 2n + 1) - n + 1]$$
$$= 2n^2 - n - [2n^2 - 4n + 2 - n + 1]$$
$$= 2n^2 - n - [2n^2 - 5n + 3]$$
$$= 2n^2 - n - 2n^2 + 5n - 3$$
$$= 4n - 3$$

21)

The first term, u_1 is equal to s_1, because there is only one term to sum.

$$u_1 = s_1 = 32(1) - 1^2 = 31$$

To find the common difference, we can find the second term u_2 and use our formula $d = u_2 - u_1$

$$u_2 = s_2 - s_1 = 32(2) - 2^2 - 31 = 29$$

$$d = u_2 - u_1 = 29 - 31 = -2$$

22)

First, we need to find u_n. Recall that $u_n = s_n - s_{n-1}$.

$$u_n = s_n - s_{n-1}$$
$$u_n = 4n^2 - 2n - \left(4(n-1)^2 - 2(n-1)\right)$$
$$u_n = 4n^2 - 2n - 4(n-1)^2 + 2(n-1)$$
$$u_n = 4n^2 - 2n - 4n^2 + 8n - 4 + 2n - 2$$
$$u_n = 8n - 6$$

So, we have

$$u_2 = 8(2) - 6 = 10$$
$$u_m = 8m - 6$$
$$u_{32} = 8(32) - 6 = 250$$

To form a geometric sequence, they must have a common ratio.

$$\frac{u_2}{u_m} = \frac{u_m}{u_{32}}$$
$$\frac{10}{8m-6} = \frac{8m-6}{250}$$
$$2500 = (8m-6)^2$$
$$2500 = 64m^2 - 96m + 36$$
$$64m^2 - 96m - 2464 = 0$$
$$2m^2 - 3m - 77 = 0$$
$$(2m+11)(m-7) = 0$$

The existence of term u_m implies that $m > 0$. So, $m = 7$.

2.2 Using Formulae

2.2.1 Finding n

23)

We have the first term $a = 5$; the common ratio $r = \frac{u_2}{u_1} = \frac{10}{5} = 2$.

So, the formula for the n^{th} term is $u_n = ar^{n-1} = 5(2)^{n-1}$.

To find the number of terms, we just need to find out which term is the last term, 1280. We can just use our n^{th} term formula and work backwards.

$$1280 = u_n = 5(2)^{n-1}$$
$$256 = (2)^{n-1}$$
$$8 = n - 1$$
$$n = 9$$

There are 9 terms in the sequence.

24)

$$a = \text{first term} = 3$$
$$d = \text{common difference} = 9 - 3 = 6$$

We know the final term is 1353. We need to know what term number this corresponds to.
We say $u_n = 1353$ and if we can find n then we know which term it is. We have a formula for u_n which we can use to then work backwards and solve for n.

$$u_n = 1353$$
Let's replace u_n with its formula
$$a + (n-1)d = 1353$$
Now let's replace a and d
$$3 + (n-1)(6) = 1353$$

$$3+6n-6=1353$$
$$6n-3=1353$$
$$6n=1356$$
$$n=226$$

Now that we know we have 226 terms we can find the sum of the sequence using the formulas for s_n where $n=226$

$$S_{226}=\frac{226}{2}[2(3)+(226-1)6]=113[6+225(6)]=153228$$

25)

The multiples of 3 over 100 are $102, 105, 107, \ldots$

This is an arithmetic sequence with $a=102$, $d=105-102=3$. So, the formula is

$$u_n=102+(n-1)3=99+3n$$

We can use the formula $s_n=\frac{n}{2}(a+l)$, where l is the last term. Clearly, l has to be the biggest multiple of 3 less than 1000, which is 999. Now, we need to work out which term 999 is (so we can plug in n). We equate $999=u_n=100+3n$

$$999=99+3n$$
$$900=3n$$
$$n=300$$

999 is the $300th$ term in this sequence.

So, the sum of all multiples of 3 between 100 and 1000 is s_{300}.

$$s_n=\frac{n}{2}(a+l)$$
$$s_{300}=\frac{300}{2}(102+999)$$
$$s_{300}=\frac{300}{2}(1101)=165150.$$

2.2.2 Simultaneous Equations

26)

Here we don't know a and d, so we will need to work backwards to find them by using the formula

The fourth term of an arithmetic sequence is 3 tells us that $u_4=3$

The sum of the first 6 terms of an arithmetic sequence tells us that $s_6=27$

Let's use the u_n and s_n formulae

$u_4=3$	$s_6=27$
The formula tells us that $a+3d=3$	The formula tells us that $\frac{6}{2}[2a+5(d)]=27$

So, we have two equations:

$$a+3d=3 \quad (1)$$
$$3[2a+5(d)]=27 \quad (2)$$

Simplifying both we get

$$a+3d=3$$
$$2a+5d=9$$

Solve simultaneously

$$a=12, d=-3$$

27)

Here we don't know a and d, so we will need to work backwards to find them by using the formula

The fortieth term of an arithmetic sequence is 106 tells us that $u_{40} = 106$
The sum of the first 40 terms of an arithmetic sequence tells us that $s_{40} = 1900$

Let's use the u_n and s_n formulae

$u_{40} = 106$	$s_{40} = 1900$
The formula tells us that	The formula tells us that
$a + 39d = 106$	$\frac{40}{2}[2a + 39(d)] = 1900$

So, we have two equations:
$a + 39d = 106$ ①
$20[2a + 39(d)] = 1900$ ②

Simplifying both we get
$a + 39d = 106$
$2a + 39d = 950$

Solve simultaneously
$a = 12, d = -3$

28)

Here we don't know a and r, so we will need to work backwards to find them by using the formula.

The fourth term of a geometric sequence is 10 tells us that $u_4 = 10$
The seventh term of a geometric sequence is 80 tells us that $u_7 = 80$

Let's use the u_n formulae

$u_4 = 10$	$u_7 = 80$
The formula tells us that	The formula tells us that
$ar^3 = 10$	$ar^6 = 80$

So, we have two equations:
$ar^3 = 10$ ①
$ar^6 = 80$ ②

Solve simultaneously

Way 1: Divide Successive Terms

$$\frac{ar^6}{ar^3} = \frac{80}{10}$$
$$r^3 = 8$$
$$r = 2$$
$ar^3 = 10$ so $a(8) = 10$ so $a = 1.25$
$$s_{20} = \frac{a(1 - r^n)}{1 - r}$$
$$= \frac{1.25(1 - 2^{20})}{1 - 2}$$
$$= 1310718.75$$

Way 2: Use Substitution

$$ar^3 = 10 \text{ ①}$$
$$r^3 = \frac{10}{a}$$
$$ar^6 = 80 \text{ ②}$$
$$a(r^3)^2 = 80$$
$$a\left(\frac{10}{a}\right)^2 = 80$$
$$\frac{100}{a} = 80$$
$$80a = 100$$
$$a = 1.25$$
$$r^3 = \frac{10}{1.25}$$
$$r^3 = 8$$
$$r = 2$$

29)

Here we don't know a and r, so we will need to work backwards to find them by using the formula

The sum to infinity of a geometric sequence is 16 tells us that $s_\infty = 16$
The sum of the first four terms is 1580 tells us that $s_4 = 15$

Let's use the s_∞ and s_n formulae

$s_\infty = 16$	$s_4 = 15$
The formula tells us that $\frac{a}{1-r} = 16$	The formula tells us that $\frac{a(1-r^4)}{1-r} = 15$

So, we have two equations:

$$\frac{a}{1-r} = 16 \; (1)$$

$$\frac{a(1-r^4)}{1-r} = 15 \; (2)$$

Solve simultaneously

Way 1: Divide Successive Terms	**Way 2: substitution re-arrange for $\frac{a}{1-r}$**	**Way 3: substitution re-arrange for a**	**Way 4: substitution re-arrange for $1-r$**
$\frac{a(1-r^4)}{1-r} \div \frac{a}{1-r} = \frac{15}{16}$ $\frac{a(1-r^4)}{1-r} \times \frac{1-r}{a} = \frac{15}{16}$ Cross cancel $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$	From (2), we have $\frac{a}{1-r}(1-r^4) = 15$ But from (1), we know $\frac{a}{1-r} = 16$ We can substitute this into (2). $16(1-r^4) = 15$ $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ Sub back into either equation to find a (choose less complicated one). $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$	We can re-arrange (1) for a to get $a = 16(1-r)$. Substitute this into (2). $\frac{16(1-r)(1-r^4)}{1-r} = 15$ Cancel $(1-r)$. $16(1-r^4) = 15$ $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ Sub back into either equation to find a (choose less complicated one). $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$	We can re-arrange (1) for a to get $1-r = \frac{a}{16}$. Substitute this into (2). $\frac{a(1-r^4)}{\frac{a}{16}} = 15$ $16(1-r^4) = 15$ $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ Sub back into either equation to find a (choose less complicated one). $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$

30)

Here we don't know a and d, so we will need to work backwards to find them by using the formula.

The sum of the first sixteen term the arithmetic sequence is 212 tells us that $s_{16} = 212$
The fifth term the arithmetic sequence is 8 tells us that $u_5 = 8$

Let's use the u_n and the s_n formulae

$s_{16} = 212$	$u_5 = 8$
The formula tells us that $\frac{16}{2}(a+15d) = 8a + 120d = 212$	The formula tells us that $a + 4d = 8$

So, we have two equations:
$$8a + 120d = 212 \text{ (1)}$$
$$a + 4d = 8 \text{ (2)}$$

Solve simultaneously. Times (2) by 8 and subtract from (1).
$$88d = 148$$
$$d = \frac{37}{22}$$
$$a = \frac{14}{11}$$

31) The sum of the first three terms of a geometric sequence is 62.755, and the sum of the infinite sequence is 440. Find the common ratio.

Here we don't know a and r, so we will need to work backwards to find them by using the formula

The sum to infinity of a geometric sequence is 440 tells us that $s_\infty = 440$
The sum of the first three terms is 62.755 tells us that $s_3 = 62.755$

Let's use the s_∞ and s_n formulae

$s_\infty = 440$	$s_3 = 62.755$
The formula tells us that $\frac{a}{1-r} = 440$	The formula tells us that $\frac{a(1-r^3)}{1-r} = 62.755$

So, we have two equations:
$$\frac{a}{1-r} = 440 \text{ (1)}$$
$$\frac{a(1-r^3)}{1-r} = 62.755 \text{ (2)}$$

Solve simultaneously

Way 1: Divide Successive Terms	**Way 2: substitution re-arrange for $\frac{a}{1-r}$**	**Way 3: substitution re-arrange for a**	**Way 4: substitution re-arrange for $1-r$**
$\frac{\frac{a(1-r^3)}{1-r}}{62.755} \div \frac{\frac{a}{1-r}}{440} =$ $\frac{\frac{a(1-r^3)}{1-r}}{62.755} \times \frac{\frac{1-r}{a}}{440} =$ Cross cancel $1-r^3 = \frac{62.755}{440}$ $r^3 = \frac{6859}{8000}$	From (2), we have $\frac{a}{1-r}(1-r^3) = 62.755$ But from (1), we know $\frac{a}{1-r} = 440$	We can re-arrange (1) for a to get $a = 440(1-r)$. Substitute this into (2). $\frac{440(1-r)(1-r^4)}{1-r} = 62.755$	We can re-arrange (1) for a to get $1-r = \frac{a}{440}$. Substitute this into (2). $\frac{a(1-r^4)}{\frac{a}{440}} = 62.755$ $440(1-r^3) = 62.755$

	$r = \sqrt[3]{\frac{6859}{8000}} = \frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r} = 440$ $\frac{a}{1-\frac{19}{20}} = 440 \Rightarrow a = 440\left(\frac{1}{20}\right) = 22$ So $a = 22, r = 19/20$.	We can substitute this into ②. $440(1-r^3) = 62.755$ $1 - r^3 = \frac{62.755}{440}$ $r^3 = \frac{6859}{8000}$ $r = \sqrt[3]{\frac{6859}{8000}} = \frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r} = 440$ $\frac{a}{1-\frac{19}{20}} = 440 \Rightarrow a =$ $440\left(\frac{1}{20}\right) = 22$ So $a = 22, r = 19/20$.	Cancel $(1-r)$. $440(1-r^3) = 62.755$ $1 - r^3 = \frac{62.755}{440}$ $r^3 = \frac{6859}{8000}$ $r = \sqrt[3]{\frac{6859}{8000}} = \frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r} = 440$ $\frac{a}{1-\frac{19}{20}} = 440 \Rightarrow a =$ $440\left(\frac{1}{20}\right) = 22$ So $a = 22, r = 19/20$.	$1 - r^3 = \frac{62.755}{440}$ $r^3 = \frac{6859}{8000}$ $r = \sqrt[3]{\frac{6859}{8000}} = \frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r} = 440$ $\frac{a}{1-\frac{19}{20}} = 440 \Rightarrow a =$ $440\left(\frac{1}{20}\right) = 22$ So $a = 22, r = 19/20$.

32)

Here we don't know a and r, so we will need to work backwards to find them by using the formula.

The sum of first two terms of a geometric sequence is 6 tells us that $s_2 = 6$
The sum of first four terms of a geometric sequence is 30 tells us that $s_4 = 30$

Let's use the u_n formulae

$s_2 = 6$	$s_4 = 30$
The formula tells us that $\frac{a(1-r^2)}{1-r} = 6$	The formula tells us that $\frac{a(1-r^4)}{1-r} = 30$

So, we have two equations:

$$\frac{a(1-r^2)}{1-r} = 6 \text{ ①}$$

$$\frac{a(1-r^4)}{1-r} = 30 \text{ ②}$$

Solve simultaneously

Way 1: Divide Successive Terms	**Way 2: Use Substitution**
$\frac{a(1-r^2)}{1-r} \div \frac{a(1-r^4)}{1-r} = \frac{6}{30}$ $\frac{a(1-r^2)}{1-r} \times \frac{1-r}{a(1-r^4)} = \frac{1}{5}$ Cross cancelling, we get $\frac{1-r^2}{1-r^4} = \frac{1}{5}$ We can use the difference of two squares to factorise the denominator. $\frac{(1-r^2)}{(1-r^2)(1+r^2)} = \frac{1}{5}$ $\frac{1}{1+r^2} = \frac{1}{5}$ $1 + r^2 = 5$	$\frac{a(1-r^2)}{1-r} = 6$ ① $a = \frac{6(1-r)}{1-r^2}$ $\frac{a(1-r^4)}{1-r} = 30$ ② $\frac{\frac{6(1-r)}{1-r^2}(1-r^4)}{1-r} = 30$ $\frac{6(1-r)(1-r^4)}{(1-r^2)(1-r)} = 30$ $\frac{6(1-r^4)}{1-r^2} = 30$ $\frac{1-r^4}{1-r^2} = 5$ We can use the difference of two squares to factorise the numerator.

	$r = \pm 2$ Sub it back into 1 (the slightly easier option) we get $\frac{a(1-2^2)}{1-2} = 6$ $a\left(\frac{-3}{-1}\right) = 6$ $a = 2$ And $\frac{a(1-(-2)^2)}{1-(-2)} = 6$ $a\left(\frac{-3}{3}\right) = 6$ $a = -6$ So, we have $r = 2, a = 2$ and $r = -2, a = -6$	$\frac{(1-r^2)(1+r^2)}{1-r^2} = 5$ $1 + r^2 = 5$ $r = \pm 2$ Sub it back into 1 (the slightly easier option) we get $\frac{a(1-2^2)}{1-2} = 6$ $a\left(\frac{-3}{-1}\right) = 6$ $a = 2$ And $\frac{a(1-(-2)^2)}{1-(-2)} = 6$ $a\left(\frac{-3}{3}\right) = 6$ $a = -6$ So, we have $r = 2, a = 2$ and $r = -2, a = -6$	

2.2.3 With Inequalities

33)

Hint: we need to solve $u_n > 0$ for n since the greatest possible sum of when we add the positive terms. Adding the terms once the sequence becomes negative will only decrease the sum and hence won't give us the greatest possible sum.

$$a = 85, d = -7$$

Here we have a **negative common difference** so greatest possible sum will be before we get any negative terms i.e. we only want to add positive terms so $u_n > 0$

$$a + (n-1)d > 0$$
$$85 - 7n + 7 > 0$$
$$92 - 7n > 0$$
$$-7n > -92$$
$$n < 13.14$$

First term that satisfies this is $n = 13$

So 13 is the most number of terms we can have before the terms become negative and hence lessen the sum

So the greatest possible sum is the sum of the first 13 terms

$$s_{13} = \frac{12}{2}[2(85) + 12(-7)] = 559$$

34)

i.

We have $u_3 = 1407, u_{10} = 1183$.

We use the formula $u_n = a + (n-1)d$

$$1407 = a + 2d \quad (1)$$
$$1183 = a + 9d \quad (2)$$

Let's solve these equations simultaneously. Subtracting 1 from 2,

$$-224 = 7d$$
$$d = -32$$

Substitute $d = -32$ back into 1.

$$1407 = a - 2(32)$$
$$a = 1471$$

ii.

The common difference is negative, so each successive term will decrease. We want to find the biggest n such that $u_n > 0$.

$$u_n = 1471 - 32(n-1) > 0$$
$$1471 - 32n + 32 > 0$$
$$1503 > 32n$$
$$46.96875 > n$$

The greatest n satisfying this constraint is $n = 46$, so there are 46 positive terms.

35)

We have $a = 1000, d = -6$. The sum of first n terms is negative. So,

$$s_n < 0$$

Using the formula for s_n

$$\frac{n}{2}\left(2(1000) + (n-1)(-6)\right) < 0$$
$$\frac{n}{2}(2000 - 6n + 6) < 0$$
$$n(1003 - 3n) < 0$$
$$n < 0 \text{ } or \text{ } n > \frac{1003}{3}$$

Since n has to be a positive integer, $n > \frac{1003}{3} \approx 334.333$.

The least n satisfying this inequality is 335.

36)

i.

We have $u_5 = 16, u_{10} = 30$.

We use the formula $u_n = a + (n-1)d$

$$16 = a + 4d \text{ (1)}$$
$$30 = a + 9d \text{ (2)}$$

Let's solve these equations simultaneously. Subtracting (1) from (2),

$$14 = 5d$$
$$d = \frac{14}{5}$$

Substitute $d = \frac{14}{5}$ back into (1).

$$16 = a + 4\left(\frac{14}{5}\right)$$
$$a = \frac{24}{5}$$

ii.

We want $s_n > 1000$. Let's use the formula

$$\frac{n}{2}(2a + (n-1)d) > 1000$$
$$\frac{n}{2}\left(2\left(\frac{24}{5}\right) + (n-1)\left(\frac{14}{5}\right)\right) > 1000$$
$$\frac{n}{2}\left(\frac{48}{5} + \frac{14}{5}n - \frac{14}{5}\right) > 1000$$
$$\frac{n}{2}\left(\frac{34}{5} + \frac{14}{5}n\right) > 1000$$
$$\frac{34}{10}n + \frac{14}{10}n^2 > 1000$$
$$7n^2 + 17n - 5000 > 0$$

Using the quadratic formula,

$$x < \frac{-17 - \sqrt{140289}}{14} \approx -27.97 \text{ } or \text{ } x > \frac{-17 + \sqrt{140289}}{14} \approx 25.54$$

Since x cannot be negative, $x > 25.54$. You need at least 26 terms for the sum to exceed 1000.

37)

This is an arithmetic sequence. We have $a = 4, d = 9 - 4 = 5$.

We want $s_n > 2000$. Let's use the formula.

$$\frac{n}{2}\left(2(4) + (n-1)(5)\right) > 2000$$
$$\frac{n}{2}(8 + 5n - 5) > 2000$$
$$\frac{n}{2}(3 + 5n) > 2000$$
$$\frac{3}{2}n + \frac{5}{2}n^2 - 2000 > 0$$
$$5n^2 + 3n - 4000 > 0$$

Using the quadratic formula,

$$n < \frac{-3 - \sqrt{80009}}{10} \text{ or } n > \frac{-3 + \sqrt{80009}}{10}$$

Since n cannot be negative, $n > \frac{-3+\sqrt{80009}}{10} \approx 27.99$. The least n satisfying this inequality is 28.

2.2.4 Worded Questions

38)

i.

If she earns 500 more every year, it'll take her $(25000 - 20000)/500 = 10$ years to reach her maximum salary. So, for the first ten years the amount she earns every year is an arithmetic sequence with initial earning $a = 20000$, and the common difference $d = 500$.

So the total she ears in the first n years must be $s_n = \frac{n}{2}(2a + (n-1)d), n \leq 10$.

$$s_{10} = \frac{10}{2}(2(20000) + (10-1)500)$$
$$s_{10} = 222500$$

The total she earns is £222500.

ii.

Over the next five years Carols has no salary rise, so she earns 25000 every year for five more years.

$$222500 + 5 \times 25000 = 347500$$

The total she earns over 15 years is £347500

iii.

It is unlikely that her salary will increase by the same amount each year.

39)

i.

This amount of mass the company extracts per year starting from 2018 is a geometric sequence, with starting value $a = 4500$ and common ratio 0.98.

The total mass extracted between 2018 and 2040, inclusive, is the sum of the first $2040 - 2018 + 1 = 23$ terms, ie s_{23}.

$$s_{23} = \frac{4500(1 - 0.98^{23})}{1 - 0.98} = 83621.86152 \ldots \approx 83600 \ tonnes$$

ii.

The mass of the mineral extracted in 2040 is simply the $23rd$ term of the geometric sequence, by our previous calculation.

$$u_{23} = 4500(0.98)^{23-1} = 2885.268312 \approx 2890 \ tonnes$$

iii.

We can see that in every year between 2018 and 2040, the company extracts more than 1500 tonnes of mineral, since even in 2040 the company is still extracting $2890 > 1500$ tonnes.

So, we know that the company paid £800 per tonne for 1500 tonnes every year for 23 years, and the rest are paid in £600 per tonne.

$$800 \times (1500 \times 23) + 600 \times (83621.86152 - 1500 \times 23) = 57073116.91$$
$$x \approx 57100000$$

40)

i.

In Model A, we have an arithmetic sequence that models the amount of batteries made per year.
We have $a = u_1 = 2600$, $u_{10} = 12000$. We can use $u_n = a + (n-1)d$ to work out the common difference.

$$u_{10} = 12000 = 2600 + (10-1)d$$
$$9400 = 9d$$
$$d = 9400/9$$

So, the number of batteries the company makes in year 2 would be
$u_2 = 2600 + (2-1)\left(\frac{9400}{9}\right) = 3644.44444 \approx 3644$ under model A.
Note: accept 3645 also

ii.

In Model B, we have an geometric sequence that models the amount of batteries made per year. We have $a = v_1 = 2600$, $v_{10} = 12000$. We can use $v_n = ar^{n-1}$ to work out the common ratio.

$$v_{10} = 12000 = 2600r^{10-1}$$
$$\frac{60}{13} = r^9$$
$$r = \sqrt[9]{\frac{60}{13}}$$

So, the number of batteries the company makes in year 2 would be $v_2 = 2600\left(\sqrt[9]{\frac{60}{13}}\right)^{2-1} = 3081.585526\ldots \approx 3080$ batteries under model B.

iii.

We calculate $s_{10} = u_1 + u_2 + \cdots + u_{10}$ using the formula $s_n = \frac{n}{2}(a + l)$.

$$s_{10} = \frac{10}{2}(2600 + 12000) = 73000$$

We calculate $t_{10} = v_1 + v_2 + \cdots + v_{10}$ using the formula $t_n = \frac{a(1-r^n)}{1-r}$

$$t_{10} = \frac{2600\left(1 - \left(\sqrt[9]{\frac{60}{13}}\right)^{10}\right)}{1 - \sqrt[9]{\frac{60}{13}}} = 62749.03356$$

The difference is

$$73000 - 62749.03356 = 10250.96644 \approx 10300$$

Note: Accept 10,200 batteries also

41)

i.

This arithmetic sequence has $a = u_1 = 140$, and $d = 160 - 140 = 20$.
Plugging into the formula

$$u_{20} = 140 + (20-1)20 = 520$$

ii.

Using the formula $s_n = \frac{n}{2}(a + l)$

$$s_{20} = \frac{20}{2}(140 + 520) = 6600$$

iii.

We can use the formula

$$s_n = \frac{n}{2}(a + l)$$
$$8500 = \frac{n}{2}(300 + 700)$$

$$8500 = \frac{n}{2}1000$$
$$8500 = 500n$$
$$n = 17$$

2.2.4.1 Being Careful with difference between years

42)

i.

The salary each year is an arithmetic sequence.
We have $a = 17000, d = 1500$. We want to solve for k, where $u_k = 32000$

$$32000 = u_k = a + (k-1)d = 17000 + (k-1)1500$$
$$15000 = (k-1)1500$$
$$10 = k - 1$$
$$k = 11$$

ii.

For the first 11 years, Jess has earned s_{11} pounds.

$$s_{11} = \frac{11}{2}(17000 + 32000) = \ 269500$$

For the last 9 years, Jess ears £32000 each year.

The total is $269500 + 32000 \times 9 = £557500$.

43)

i.

John gets a gift worth £60 on his tenth birthday, £75 on his eleventh, £90 on his twelfth. So, the total he gets is

$$£60 + £75 + £90 = £225$$

ii.

The value of the gift each year is an arithmetic sequence.
We have $a = 60$, $d = 15$.
The eighteenth birthday is the $18 - 10 + 1 = 9th$ year he receives gifts from his uncle.

$$u_9 = a + (9-1)d$$
$$u_9 = 60 + 8(15) = £180.$$

iii.

The twenty-first birthday is the $21 - 10 + 1 = 12th$ year he receives gifts from his uncle.
The total value of gifts is s_{12}

$$s_{12} = \frac{12}{2}\big(2(60) + (12-1)(15)\big) = \ £1710$$

iv.

We have $s_n = 3375$

$$3375 = s_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}\big(2(60) + (n-1)(15)\big)$$
$$3375 = \frac{n}{2}(120 + 15n - 15)$$
$$3375 = \frac{n}{2}(105 + 15n)$$
$$3375 = \frac{105}{2}n + \frac{15}{2}n^2$$
$$225 = \frac{7}{2}n + \frac{1}{2}n^2$$
$$n^2 + 7n = 450 = 25 \times 18$$

v.

We have $n^2 + 7n = 25 \times 18$

$$n(n+7) = 25 \times 18$$

We deduce that

$$n = 18 \ or - 25$$

Clearly $n = 18$ is the only answer that makes sense here. This means he was 27 years old.

44)

i.

The number of mobile phones produced each week forms an arithmetic sequence. We have $a = 200, d = 20$.

We solve for N, where $u_N = 600$.

$$600 = u_N = 200 + (N-1)20$$
$$400 = (N-1)20$$
$$N - 1 = 20$$
$$N = 21$$

ii.

Hint: Find s_{21} and then from week 22 to 52 i.e. 31 weeks multiply by 600

For the first 21 weeks, the company makes s_{21} phones in total.

$$s_{21} = \frac{21}{2}(200 + 600) = 8400$$

Then for the next $52 - 21 = 31$ weeks, the company makes 600 phones each week. The total produced is therefore

$$8400 + 31 \times 600 = 27000$$

45)

i.

Hint: This is the $13th$ year $(2002 - 1989 \text{ or } 2002 - 1989 + 1)$

The number of bicycles produced each week forms an arithmetic sequence. We have $a = 140, u_{12} = 206$.

We can solve for d using the formula for u_n.

$$206 = u_{12} = 140 + (12 - 1)d$$
$$66 = 11d$$
$$d = 6$$

i.

During the first 12 weeks, the company produced s_{12} bikes.

$$s_{12} = \frac{12}{2}(140 + 206) = 2076$$

For the remaining $52 - 12 = 10$ weeks, the company produced 206 bikes per week every week. The total is therefore

$$2076 + 206 \times 10 = 4136$$

46)

i.

We have $a = 160, r = \frac{240}{160} = \frac{3}{2}$. Since 2002 is the $2002 - 1990 + 1 = 13th$ year, we calculate u_{13}

$$u_{13} = ar^{13-1} = 160\left(\frac{3}{2}\right)^{12} = 20759.41406 \approx 20759$$

ii.

We calculate $u_n > 5000$.

$$5000 < u_n = 160\left(\frac{3}{2}\right)^{n-1}$$
$$5000 < 160\left(\frac{3}{2}\right)^{n-1}$$
$$\frac{125}{4} < \left(\frac{3}{2}\right)^{n-1}$$

We take log of both sides, then divide by $\ln\left(\frac{3}{2}\right) > 0$, so we don't need to reverse the inequality.

$$\ln\frac{125}{4} < (n-1)\ln\left(\frac{3}{2}\right)$$
$$\frac{\ln\left(\frac{125}{4}\right)}{\ln\left(\frac{3}{2}\right)} < n - 1$$
$$n < 9.489 \ldots$$

So, we have $n = 10$ is the year units sold first exceeds 5000.

Which is year $1990 + 10 - 1 = 1999$.

iii.

We already calculated 2002 is the $13th$ year. So, this value is just s_{13}. We use the formula for geometric series.

$$s_{13} = \frac{160\left(1 - \left(\frac{3}{2}\right)^{13}\right)}{1 - \left(\frac{3}{2}\right)} = 61958.24219 \approx 61958$$

3 Gold

3.1 Without Formulae

47)

i.

We use the fact that they have a common ratio.

$$\frac{\frac{2k+3}{4}}{\frac{k-3}{2}} = \frac{\frac{12k+3}{8}}{\frac{2k+3}{4}}$$

$$\frac{2k+3}{4} \times \frac{2k+3}{4} = \frac{12k+3}{8} \times \frac{k-3}{2}$$

$$16(2k+3)^2 = 16(12k+3)(k-3)$$

$$(2k+3)^2 = (12k+3)(k-3)$$

$$4k^2 + 12k + 9 = 12k^2 - 33k - 9$$

$$8k^2 - 45k - 18 = 0$$

ii.

We solve for k.

$$(8k+3)(k-6) = 0$$

$$k = -\frac{3}{8} \text{ or } 6$$

We split into two cases.

Case $k = -\frac{3}{8}$	**Case $k = 6$**
The sequence becomes $-\frac{27}{16}, \frac{9}{16}, -\frac{3}{16}$ We have common ratio $r = \frac{\frac{9}{16}}{-\frac{27}{16}} = -\frac{1}{3}$ So, the first 5 terms are $-\frac{27}{16}, \frac{9}{16}, -\frac{3}{16}, \frac{1}{16}, -\frac{1}{48}$	The sequence becomes $\frac{3}{2}, \frac{15}{4}, \frac{75}{8}$ We have common ratio $r = \frac{\frac{15}{4}}{\frac{3}{2}} = \frac{5}{2}$ So, the first 5 terms are $\frac{3}{2}, \frac{15}{4}, \frac{75}{8}, \frac{375}{16}, \frac{1875}{32}$

48)

If $p, q, 10$ are in arithmetic progression, then they must have a common difference.

$$q - p = 10 - q$$

$$q = \frac{10+p}{2} \quad (1)$$

If $q, p, 10$ are in geometric progression, then they must have a common ratio.

$$\frac{p}{q} = \frac{10}{p}$$

$$p^2 = 10q \quad (2)$$

We can solve (1) and (2) simultaneously. We substitute (1) into (2).

$$p^2 = \frac{10(10+p)}{2}$$

$$p^2 = 50 + 5p$$

$$p^2 - 5p - 50 = 0$$

$$(p-10)(p+5) = 0$$

$$p = 10 \ or - 5$$

If we substitute these two values of p back into (1), we get $p = 10, q = 10$ or $p = -5, q = \frac{5}{2}$. Since p, q are distinct numbers, $p = -5, q = \frac{5}{2}$.

3.2 Using Formulae

3.2.1 Simultaneous Equations

49)

We have $u_2 + u_3 = -12, u_3 + u_4 = -36$. Using the formula $u_n = ar^{n-1}$, we get

$$ar + ar^2 = -12 \quad (1)$$
$$ar^2 + ar^3 = -36 \quad (2)$$

Dividing (1) by (2) gives (we can do this because neither is 0). Note you can also factorise each and then use substitution instead.

$$\frac{ar + ar^2}{ar^2 + ar^3} = \frac{-12}{-36}$$
$$\frac{r + r^2}{r^2 + r^3} = \frac{1}{3}$$
$$\frac{r(1+r)}{r^2(1+r)} = \frac{1}{3}$$
$$\frac{1}{r} = \frac{1}{3}$$
$$r = 3$$

50)

We have $u_1 + u_2 = 8,\ u_3 + u_4 = 2$. Using the formula $u_n = ar^{n-1}$we get

$$a + ar = 8 \quad (1)$$
$$ar^2 + ar^3 = 2 \quad (2)$$

Dividing (1) by (2) gives (we can do this because neither is 0)

$$\frac{a + ar}{ar^2 + ar^3} = \frac{8}{2}$$
$$\frac{1 + r}{r^2 + r^3} = 4$$
$$\frac{(1+r)}{r^2(1+r)} = 4$$
$$\frac{1}{r^2} = 4$$
$$r = \pm\frac{1}{2}$$

Substituting $r = \pm\frac{1}{2}$ back into ① gives

$$a + \frac{a}{2} = 8$$
$$\frac{3}{2}a = 8$$
$$a = \frac{16}{3}$$
$$a - \frac{a}{2} = 8$$
$$\frac{1}{2}a = 8$$
$$a = 16$$

So, $r = \frac{1}{2}, a = \frac{16}{3}$ or $r = -\frac{1}{2}, a = 16$.

51)

We have $u_1 + u_2 = 50,\ u_2 + u_3 = 30$. Using the formula $u_n = ar^{n-1}$ we get

$$a + ar = 50 \text{ ①}$$
$$ar + ar^2 = 30 \text{ ②}$$

Dividing ① by ② gives (we can do this because neither is 0)

$$\frac{a + ar}{ar + ar^2} = \frac{50}{30}$$
$$\frac{1 + r}{r + r^2} = \frac{5}{3}$$
$$\frac{(1 + r)}{r(1 + r)} = \frac{5}{3}$$
$$\frac{1}{r} = \frac{5}{3}$$
$$r = \frac{3}{5}$$

Substituting $r = \frac{3}{5}$ back into ① gives

$$a + \frac{3}{5}a = 50$$
$$\frac{8}{5}a = 50$$
$$a = \frac{250}{8} = \frac{125}{4}$$

Since $-1 < r = \frac{3}{5} < 1$, we can substitute a, r into the formula for sum to infinity to get

$$s_\infty = \frac{a}{1 - r} = \frac{\frac{125}{4}}{1 - \frac{3}{5}} = \frac{625}{8}$$

52)

We know their sum to infinities equal, so let's equate the two sums to infinities.

$$\frac{3a}{1 - r} = \frac{a}{1 - (-2r)}$$
$$(1 + 2r)3a = a(1 - r)$$

Divide both sides by a. This assumes that $a \neq 0$, if this is the case then both geometric sequences would be just $0, 0, 0, \ldots$ then any common ratio suffices.

$$3(1 + 2r) = 1 - r$$
$$3 + 6r = 1 - r$$
$$7r = -2$$
$$r = -\frac{2}{7}$$

53)

We know the formula $s_n = \frac{a(1-r^n)}{1-r}$, let's substitute this into $s_{10} = 4s_5$.

$$\frac{a(1-r^{10})}{1-r} = \frac{4a(1-r^5)}{1-r}$$

Cancelling terms out, we get

$$1 - r^{10} = 4(1-r^5)$$
$$1 - r^{10} = 4 - 4r^5$$
$$r^{10} - 4r^5 + 3 = 0$$

This is a hidden quadratic, set $x = r^5$.

$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 3 \text{ or } 1$$

Since $r \neq 1$ (because our formula for geometric series would end up dividing by 0), $r = \sqrt[5]{3}$.

54)

We have the fourth term, ar^3, is eight times the first term, a.

$$ar^3 = 8a$$
$$r^3 = 8$$
$$r = 2$$

We also have $s_{10} = 2557.5$

$$2557.5 = \frac{a(1-r^{10})}{1-r} = \frac{a(1-2^{10})}{1-2}$$
$$2557.5 = 1023a$$
$$a = \frac{5}{2}$$

55)

We have $u_1 = u_2 + 9, s_\infty = 81$

Let's use the formula for u_n and s_∞

$s_\infty = 81$ The formula tells us that $\frac{a}{1-r} = 81$	$u_1 = u_2 + 9$ The formula tells us that $a = ar + 9$

We have

$$\frac{a}{1-r} = 81 \quad (1)$$
$$a = ar + 9 \quad (2)$$

Rearranging (1), we get $a = 81(1-r)$. Sub it into (2).

$$81(1-r) = 81(1-r)r + 9$$
$$81 - 81r = 81r - 81r^2 + 9$$
$$81r^2 - 162r + 72 = 0$$
$$9r^2 - 18 + 8 = 0$$
$$(3r-4)(3r-2) = 0$$
$$r = \frac{4}{3} or \frac{2}{3}$$

But this is an infinite geometric series, so $-1 < r < 1$. Therefore, $r = \frac{2}{3}$.

56)

We have $u_8 = 3u_3$. Let's use the formula for u_n.

$$a + 7d = 3(a + 2d)$$
$$a + 7d = 3a + 6d$$
$$2a = d \ (*)$$

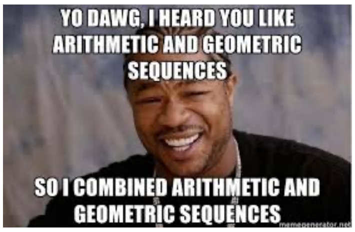
Let's calculate the sum of first eight terms, s_8, and the sum of first four terms s_4.

s_8	s_4

$s_8 = \frac{8}{2}(2a + 7d)$ $s_8 = 4(2a + 7d)$ $s_8 = 8a + 28d$ Let's substitute the equation $(*)$ in. $s_8 = 8a + 28(2a)$ $s_8 = 64a$	$s_4 = \frac{4}{2}(2a + 3d)$ $s_4 = 2(2a + 3d)$ $s_4 = 4a + 6d$ Let's substitute the equation $(*)$ in. $s_4 = 4a + 6(2a)$ $s_4 = 16a$

We have $64a = 4(16a)$, ie $s_8 = 4s_4$.

3.2.2 Arithmetic and Geometric Together In One Question



57)

i.

Let the geometric sequence by u_n, the arithmetic sequence be v_n.
We have

$$u_1 = v_1 = 8 = a$$

Both series have the same first term, so they can have a common a. Then, we have

$$u_2 = v_9$$
$$u_3 = v_{21}$$

We substitute in the formula for geometric sequence and arithmetic sequence on each side.

$$8r = 8 + 8d \text{ (1)}$$
$$8r^2 = 8 + 20d \text{ (2)}$$

Let's solve these equations simultaneously.
We simplify and rearrange (1) to get

$$r = 1 + d$$
$$d = r - 1 \text{ (3)}$$

We substitute this into (2).

$$8r^2 = 8 + 20(r - 1)$$
$$8r^2 = 8 + 20r - 20$$
$$8r^2 - 20 + 12 = 0$$
$$2r^2 - 5 + 3 = 0$$
$$(2r - 3)(r - 1) = 0$$
$$r = \frac{3}{2} or\ 1$$

We're told that $r \neq 1$, so $r = \frac{3}{2}$.

ii.

The geometric sequence: $u_4 = ar^3 = 8\left(\frac{3}{2}\right)^3 = 27$

For the arithmetic sequence, we need to work out the common difference d. We use equation (3) $d = r - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

The arithmetic sequence: $v_4 = a + 3d = 8 + 3\left(\frac{1}{2}\right) = \frac{19}{2}$.

58)

i.

Let the geometric sequence by u_n, the arithmetic sequence be v_n.
We have

$$u_1 = v_1 = 2 = a$$

Both series have the same first term, so they can have a common a. Then, we have

$$u_2 = v_2$$
$$u_3 = v_5$$

We substitute in the formula for geometric sequence and arithmetic sequence on each side.

$$2r = 2 + d \; ①$$
$$2r^2 = 2 + 4d \; ②$$

Let's solve these equations simultaneously.

We rearrange ① to get

$$r = 1 + \frac{d}{2} \; ③$$

We substitute this into ②.

$$2r^2 = 2 + 4d$$
$$2\left(1 + \frac{d}{2}\right)^2 = 2 + 4d$$
$$2 + 2d + \frac{d^2}{2} = 2 + 4d$$
$$\frac{d^2}{2} - 2d = 0$$
$$d^2 - 4d = 0$$
$$d(d-4) = 0$$
$$d = 0 \text{ or } 4$$

We may assume that $d \neq 0$. So, $d = 4$

ii.

We substitute d back into equation ③.

$$r = 1 + \frac{4}{2} = 3$$

59)

i.

We have $a = u_1 = 27$.

We also have the sum to infinity $s_\infty = \frac{81}{2} = \frac{a}{1-r}$

Substitute $a = 27$.

$$\frac{81}{2} = \frac{27}{1-r}$$
$$81(1-r) = 54$$
$$1 - r = \frac{54}{81}$$
$$r = 1 - \frac{54}{81} = \frac{27}{81} = \frac{1}{3}$$

ii.

We have the formula $v_n = b + (n-1)d$, where b is the first term, d is the common difference. We use b instead of a here because a is already used for the first term of the geometric sequence. We do not know if they are the same.

We have $v_2 = u_2$, $v_4 = u_4$

$$b + d = ar = 9$$
$$b + 3d = ar^3 = 1$$

Solving the equations simultaneously, we get $d = -4, b = 13$.

Now, $\sum_{n=1}^{N} v_n = s_n = \frac{n}{2}(2b + (n-1)d)$

$$s_n = \frac{n}{2}(2(13) + (n-1)(-4))$$
$$= \frac{n}{2}(26 - 4n + 4)$$
$$= \frac{n}{2}(30 - 4n)$$
$$= 15n - 2n^2$$

We want $s_n > 0$.

$$15n - 2n > 0$$

$$2n^2 - 15n < 0$$
$$n(2n - 15) < 0$$
$$0 < n < \frac{15}{2}$$

Since n has to be an integer, the greatest n is 7.

60)

i. We have, from the formula, $u_n = 1.6 + 1.5(n-1) = 1.5n + 0.1$, $v_N = 3r^{n-1}$.
So,

$$u_n - v_n = 1.5n + 0.1 - 3(1.2)^{n-1}$$

ii. If $u_n > v_n$, $u_n - v_n > 0$.

$$1.5n + 0.1 - 3(1.2)^{n-1} > 0$$

This can only be solved via trial or error or graphically
$3 \leq n \leq 9$

iii. This should be solved graphically
1.64

61)

i. Substitute the formulas for an arithmetic sequence, we have

$$a + 6d, a + 2d, a$$

Form a geometric sequence.

By the common ratio property, we have

$$\frac{a+2d}{a+6d} = \frac{a}{a+2d}$$
$$(a+2d)^2 = a(a+6d)$$
$$a^2 + 4ad + 4d^2 = a^2 + 6ad$$
$$4d^2 - 2ad = 0$$
$$2d(2d - a) = 0$$

Since the common difference d is non-zero, we have $d = \frac{a}{2}$

ii. We have the seventh term $a + 6d = 3$

$$a + 6\left(\frac{a}{2}\right) = 3$$
$$4a = 3$$
$$a = \frac{3}{4}$$

Substitute this back into $d = \frac{a}{2}$ gives us $d = \frac{3}{8}$.

We also have $r = \frac{a}{a+2d} = \frac{1}{2}$.

Now we have the sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms of the geometric sequence by 200.

$$\frac{n}{2}\left(2\left(\frac{3}{4}\right) + (n-1)\left(\frac{3}{8}\right)\right) - \frac{\left(\frac{3}{4}\right)\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \left(\frac{1}{2}\right)} \geq 200$$
$$\frac{9}{16}n + \frac{3}{16}n^2 - \frac{3}{8} + \frac{3}{8}\left(\frac{1}{2}\right)^n \geq 200$$

We can solve this using a calculator or via trial and error. We have $n \geq 31.68$. The smallest integer n is 32.

62)

Given formula for s_n, we can work out $a = u_1 = s_1$.

$$a = 4(1)^2 - 2(1) = 2$$

We can work out $u_2 = s_2 - s_1 = s_2 - a$

$$u_2 = 4(2)^2 - 2(2) - 2 = 10$$

So, $d = u_2 - u_1 = 10 - 2 = 8$.

Now we have a general formula for $u_n = a + (n-1)d = 2 + (n-1)8 = 8n - 6$.

Now, we can use the common ratio formula to find r.

$$\frac{u_m}{u_2} = r$$
$$\frac{u_{32}}{u_m} = r$$

Multiply the above together, we get

$$\frac{u_m}{u_2} \times \frac{u_{32}}{u_m} = \frac{u_{32}}{u_2} = r^2$$

We use the formula for u_n

$$\frac{8(32) - 6}{8(2) - 6} = r^2$$
$$25 = r^2$$
$$r = \pm 5$$

We know

$$\frac{u_m}{u_2} = r = \pm 5$$

So,

$$u_m = \pm 5(8(2) - 6) = \pm 50$$

We find m by plugging it back into the formula for u_n.

$$\pm 50 = 8m - 6$$

$50 = 8m - 6$ $56 = 8m$ $m = 7$	$-50 = 8m - 6$ $-44 = 8m$ $m = -\frac{11}{2}$

Since m must be a positive integer, $m = 7$.

3.2.3 Worded

63)

i. We have an arithmetic sequence with the first term A, and common difference $d + 1$. So, the formula is $u_n = A + (n - 1)(d + 1)$

Plug in $n = 14$.

$$u_{14} = A + (14 - 1)(d + 1)$$
$$u_{14} = A + 13d + 13$$

ii. Let Yi's running time be v_n. Then this arithmetic sequence has firs term $A - 13$, common difference $2d - 1$. The formula is

$$v_n = A - 13 + (n - 1)(2d - 1)$$

When $n = 14$,

$$v_{14} = A - 13 + 13(2d - 1) = A + 26d - 26$$

We are given that $u_{14} = v_{14}$

$$A + 13d + 13 = A + 26d - 26$$
$$13d + 13 = 26d - 26$$
$$39 = 13d$$
$$d = 3$$

iii. We have $s_{14} = 784$. Substituting the first term and the common difference into the equation for arithmetic series, we get

$$s_n = \frac{n}{2}\big(2A + (n - 1)(d + 1)\big)$$
$$784 = s_{14} = \frac{14}{2}\big(2A + (14 - 1)(3 + 1)\big)$$
$$784 = 7(2A + 52)$$
$$112 = 2A + 52$$
$$A = 30$$

64)

i.

We have an arithmetic sequence with initial term $a = 1500$, common difference $d = -x$. The total sales during the first six months is £8100. So $s_6 = 8100$. We substitute this value into the arithmetic series formula to find out x.

$$s_n = \frac{n}{2}(2a + (n-1)d)$$
$$s_6 = \frac{6}{2}(2(1500) + (6-1)(-x))$$
$$8100 = 3(3000 - 5x)$$
$$2700 = 3000 - 5x$$
$$-300 = -5x$$
$$x = 60$$

ii.

We substitute $a = 1500, d = -60$ into the formula for n^{th} term.

$$u_n = a + (n-1)d$$
$$u_8 = 1500 + (8-1)(-60)$$
$$= 1500 - 420$$
$$= 1080$$

The sales in the eighth month is £1080.

iii.

The expected value of sales in pounds during the first n months is just the sum s_n.

$$s_n = \frac{n}{2}(2a + (n-1)d)$$
$$s_n = \frac{n}{2}\big(2(1500) + (n-1)(-60)\big)$$
$$s_n = \frac{n}{2}(3000 - 60n + 60)$$
$$s_n = \frac{n}{2}(3060 - 60n)$$
$$s_n = \frac{n}{2}(60)(51 - n)$$
$$s_n = 30n(51 - n)$$
$$k = 30$$

iv.

This model cannot be valid for a long period of time because eventually the sales in a month would become negative, which is not possible.

65)

i.

The pay each day forms an arithmetic sequence with initial term a, common difference d.

$$u_n = a + (n-1)d$$

We know a picker earns £40.75 on their $30th$ day. $u_{30} = 40.75$

$$40.75 = a + 29d \text{ (1)}$$

ii.

We are given that a picker who works for all 30 days will earn £1005 in total, hence $s_{30} = 1005$

Since we already know the value for the $30th$ term. We can use the formula

$$s_n = \frac{n}{2}(a + l)$$
$$s_{30} = \frac{30}{2}(a + 40.75)$$
$$1005 = 15(a + 40.75) \text{ (2)}$$

iii.

We can solve a from equation (2) in the previous question.

$$67 = a + 40.75$$
$$a = 26.25$$

Put this back into (1)

$$40.75 = 26.25 + 29d$$
$$14.5 = 29d$$

$$d = \frac{1}{2}$$
$$50\,p$$

66)

i.

The amount Shelim earns each year forms an arithmetic sequence with $a = 14000$, $d = 1500$ (until he reaches £26000).

We can calculate how much she will earn in year 9 by calculating u_9, and making sure this number is not more than 26000.

$$u_9 = a + 8d$$
$$u_9 = 14000 + 8(1500) = 26000$$

So Shelim will earn £26000 in year 9, and his salary will not grow after that.

ii.

We use the formula $s_n = \frac{n}{2}(a + l)$, we can do this because the first 9 years his salary per year forces an arithmetic sequence.

$$s_9 = \frac{9}{2}(14000 + 26000) = 180000$$

The answer is £180000

iii.

Anna's salary for the first 10 years forms an arithmetic sequence v_n with starting salary A, and common difference 1000. We can work out A by using our knowledge that she earns £26000 in year 10.

$$v_n = a + (n-1)d$$
$$26000 = A + (10-1)1000$$
$$26000 = A + 9000$$
$$A = 17000$$

We label the corresponding sums of this sequence v_n to be t_n. Now, we can plug these values into the formula $t_n = \frac{n}{2}(a + l)$ to work out how much Anna will earn for the first 10 years.

$$t_{10} = \frac{10}{2}(17000 + 26000) = 215000$$

So, Anna will earn £215000 in the first 10 years.

Shelim earns £180000 in the first 9 years, he will earn another £26000 in year 10, bringing the total to £180000 + £26000 = £206000.

So, the total difference is

$$£206000 - £215000 = £9000.$$

3.2.4 With Logs

67)

We have the first term $\ln a$, and the common difference is $\ln 3$. We can use our knowledge of the $13th$ term to work out the value of a.

$$8\,ln\,9 = u_{13} = \ln a + (13-1)\ln 3$$
$$8\ln 9 = \ln a + 12\ln 3$$
$$8\,ln\,9 - 12\,ln\,3 = \ln a$$
$$ln\,9^8 - ln\,3^{12} = \ln a$$

$$\ln\left(\frac{9^8}{3^{12}}\right) = \ln a$$
$$\frac{9^8}{3^{12}} = a$$
$$a = \frac{9^8}{3^{12}} = \frac{9^8}{9^6} = 81$$

68)

i.

$$r = \frac{\log_2 x}{2\log_2 x} = \frac{1}{2}$$

ii.

We have $a = 2\log_2 x\,, r = \frac{1}{2}$

$$S_\infty = \frac{a}{1-r} = \frac{2\log_2 x}{1-\frac{1}{2}} = 4\log_2 x$$

iii.

$$d = \log_2\frac{x}{2} - \log_2 x = \log_2\left(\frac{\frac{x}{2}}{x}\right) = \log_2\frac{1}{2} = -1$$

iv.

We have $a = \log_2 x\,, d = -1$, we use the formula for S_{12}.

$$S_{12} = \frac{12}{2}\left(2\log_2 x + (12-1)(-1)\right)$$
$$S_{12} = 6(2\log_2 x - 11)$$
$$S_{12} = 12\log_2 x - 66$$

v.

We have that $S_{12} = \frac{S_\infty}{2}$. Substitute our values in.

$$12\,log_2\,x - 66 = \frac{4\,log_2\,x}{2}$$
$$12\log_2 x - 66 = 2\log_2 x$$
$$10\log_2 x = 66$$
$$\log^2 x = 6.6$$
$$x = 2^{6.6}$$

3.2.5 With Inequalities and Logs

69)

This series has $a = 3, r = \frac{6}{3} = 2$.

We want $s_n > 100000$. Let's use the formula.

$$\frac{a(r^n-1)}{r-1} > 100000$$
$$\frac{3(2^n-1)}{2-1} > 1000000$$
$$3(2^n) - 3 > 1000000$$
$$3(2^n) > 99997$$
$$2^n > \frac{99997}{3}$$
$$n > \log_2\frac{99997}{3} \approx 15.02$$

We need at least 16 terms.

70)

$u_2 = 192$

$$ar = 192$$

$u_3 = 144$

$$ar^2 = 144$$

Solving these simultaneously

$$a = \frac{192}{r}$$

$$\frac{192}{r}r^2 = 144$$

$$r = \frac{144}{192} = \frac{3}{4}$$

$$a\left(\frac{3}{4}\right) = 192$$

$$a = \frac{192}{\frac{3}{4}} = 256$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

We need to solve when the sum exceeds 1000

$$\frac{256\left(1-\frac{3^n}{4}\right)}{1-\frac{3}{4}} > 1000$$

$$\frac{256\left(1-\frac{3^n}{4}\right)}{\frac{1}{4}} > 1000$$

$$256\left(1-\frac{3^n}{4}\right) > 250$$

$$1-\frac{3^n}{4} > \frac{250}{256}$$

$$-\frac{3^n}{4} > \frac{250}{256} - 1$$

$$-\frac{3^n}{4} > -\frac{3}{128}$$

Note: we swapped the inequality sign since we divide by a negative

$$\frac{3^n}{4} < \frac{3}{128}$$

$$log\frac{3^n}{4} < log\frac{3}{128}$$

$$n\log\frac{3}{4} < log\frac{3}{128}$$

$$n > \frac{log\frac{3}{128}}{log\frac{3}{4}}$$

Note: we swapped the inequality sign since $log\frac{3}{4}$ is less than zero

$$n > 13.05$$

$$n = 14$$

71)

$$a = 120$$

$$S_\infty = \frac{120}{1-r} = 480$$

$$120 = 480(1-r)$$

$$120 = 480 - 480r$$

$$480r = 360$$

$$r = \frac{3}{4}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

We need to solve when the sum is greater than 300

$$\frac{120(1-\frac{3}{4}^n)}{1-\frac{3}{4}} > 300$$

$$\frac{120\left(1-\frac{3}{4}^n\right)}{\frac{1}{4}} > 300$$

$$120\left(1-\frac{3}{4}^n\right) > 75$$

$$1-\frac{3}{4}^n > \frac{75}{120}$$

$$-\frac{3}{4}^n > \frac{75}{120} - 1$$

$$-\frac{3}{4}^n > -\frac{3}{8}$$

Note: we swapped the inequality sign since we divide by a negative

$$\frac{3}{4}^n < \frac{3}{8}$$

$$log\frac{3}{4}^n < \log\frac{3}{8}$$

$$n\log\frac{3}{4} < \log\frac{3}{8}$$

$$n > \frac{log\frac{3}{8}}{log\frac{3}{4}}$$

Note: we swapped the inequality sign since $log\frac{3}{4}$ is less than zero

$$n > 3.409$$

$$n = 4$$

72)

i.

We have $a = u_1 = \frac{1}{81}$. We can use our formula and knowledge of u_4 to work out r.

$$\frac{1}{3} = u_4 = \frac{1}{81}r^3$$
$$27 = r^3$$
$$r = 3$$

ii.

We can use the formula for s_n.

$$\frac{a(r^n - 1)}{r - 1} > 40$$
$$\frac{\frac{1}{81}(3^n - 1)}{3 - 1} > 40$$
$$\frac{3^n - 1}{162} > 40$$
$$3^n - 1 > 6480$$
$$3^n > 6481$$
$$n > \log_3 6481 \approx 7.99$$

So, the smallest value of n satisfying the above would be $n = 8$.

73)

i.

We have $r = \frac{u_2}{u_1} = \frac{1.6}{0.64} = \frac{5}{2}$

ii.

We know that $a = u_1 = 0.64$, substitute a, r into the formula for s_n

$$s_6 = \frac{0.64\left(\left(\frac{5}{2}\right)^6 - 1\right)}{\frac{5}{2} - 1} = 103.74$$

iii.

Let's substitute the formula for s_n into the inequality and solve for n.

$$s_n > 7500$$
$$\frac{0.64\left(\left(\frac{5}{2}\right)^n - 1\right)}{\frac{5}{2} - 1} > 75000$$
$$\frac{32}{75}\left(\left(\frac{5}{2}\right)^n - 1\right) > 75000$$
$$\left(\frac{5}{2}\right)^n - 1 > 175781.25$$
$$\left(\frac{5}{2}\right)^n > 175182.25$$
$$n \ln\left(\frac{5}{2}\right) > \ln 175182.25$$

Divide both sides by $\ln\frac{5}{2}$, we don't need to reverse the inequality because $\frac{5}{2} > 1$.

$$n > \frac{\ln 175181.25}{\ln\frac{5}{2}} \approx 13.18$$

The least value of n is therefore 14.

74)

The amount of money Carlos saves each year forms a geometric sequence u_n with $a = 100, r = 1.1$.

The total money he has saved would be s_n after n years, so we want $s_n > 1000$

$$\frac{100(1 - 1.1^n)}{1 - 1.1} > 1000$$
$$-1000(1 - 1.1^n) > 1000$$
$$-1000 + 1000(1.1^n) > 1000$$
$$1000(1.1^n) > 2000$$
$$1.1^n > 2$$
$$n > \log_{1.1} 2 \approx 7.27$$

So after 8 years Carlos' saving would exceed £1000.

75)

i.

We have $a = 5, r = \frac{4}{5}, s_k > 24.95$. Let's substitute the formula for s_k.

$$\frac{5\left(1-\left(\frac{4}{5}\right)^k\right)}{1-\frac{4}{5}} > 24.95$$
$$25\left(1-\left(\frac{4}{5}\right)^k\right) > 24.95$$
$$1-\left(\frac{4}{5}\right)^k > \frac{499}{500}$$
$$\frac{1}{500} > \left(\frac{4}{5}\right)^k$$
$$\log\left(\frac{1}{500}\right) > \log\left(\left(\frac{4}{5}\right)^k\right)$$
$$\log(0.002) > k\log 0.8$$

We divide both sides by $\log 0.8$. Since $0.8 < 1$, $\log 0.8 < 0$ so we need to reverse the inequality sign.

$$k > \frac{\log(0.002)}{\log 0.8}$$

ii.

$k > \frac{\log(0.002)}{\log 0.8} \approx 27.9$. The smallest positive value of k is 28.

76)

We have $a = 20, r = \frac{7}{8}$. Let's substitute the formula for S_∞ and S_N into the inequality $S_\infty - S_N < 0.5$ and solve for N.

$$\frac{20}{1-\frac{7}{8}} - \frac{20\left(1-\left(\frac{7}{8}\right)^N\right)}{1-\frac{7}{8}} < 0.5$$
$$160 - 160\left(1-\left(\frac{7}{8}\right)^N\right) < 0.5$$
$$160 - 160 + 160\left(\frac{7}{8}\right)^N < 0.5$$
$$\left(\frac{7}{8}\right)^N < \frac{1}{320}$$

When taking log of base that is less than one, we need to reverse the inequality sign.

$$N > \log_{\frac{7}{8}}\frac{1}{320} \approx 43.198$$

Therefore, the smallest value of N is 44.

77)

i.

We have $a = 25000$,$r = 1.03$. We calculate the population at the end of year 2, $u_2 = ar = 25000(1.03) = 25750$

ii.

1.03

iii.

We are trying to solve $u_N > 40000$. Let's substitute the formula.

$$ar^{N-1} > 40000$$
$$25000(1.03)^{N-1} > 40000$$
$$1.03^{N-1} > 1.6$$
$$\log 1.03^{N-1} > \log 1.6$$

$$(N-1)\log 1.03 > \log 1.6$$

iv.

We continue from the previous part, dividing both sides by $\log 1.03$. Since $1.03 > 1$, $\log 1.03 > 0$.

$$N-1 > \frac{\log 1.6}{\log 1.03}$$
$$N > 1 + \frac{\log 1.6}{\log 1.03} \approx 16.9$$

Therefore $N = 17$

v.

Each year the amount given to charity is equal to u_n. So over 10 years this value will be s_n

$$s_n = \frac{a(1-r^{10})}{1-r} = \frac{25000(1-1.03^{10})}{1-1.03} = 286596.98 \approx £287000$$

3.2.6 With Trig

78)

This geometric progression has $a = 1$, $r = \frac{2\cos^2\theta}{1} = 2\cos^2\theta$.

To have an infinite sum, $-1 < r < 1$, but $2\cos^2\theta \geq 0$. So, we only need to make sure

$$2\cos^2\theta < 1$$
$$\cos^2\theta < \frac{1}{2}$$
$$-\frac{1}{\sqrt{2}} < \cos\theta < \frac{1}{\sqrt{2}}$$

Note: $2\cos^2\theta$ is always greater than or equal to zero so we could have also started by finding where $2\cos^2\theta = 1$ and used this to solve

We need to consider the interval $-\pi < \theta < \pi$:
We know $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so in the interval $-\pi < \theta < \pi$, we can deduce that

$$-\frac{3\pi}{4} < \theta < \frac{\pi}{4} \quad or \quad \frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

i.

If θ is in the above region, we can calculate the sum to infinity using the formula $s_\infty = \frac{a}{1-r}$

$$s_\infty = \frac{a}{1-r} = \frac{1}{1-2\cos^2\theta} = -\frac{1}{2\cos^2\theta - 1} = -\frac{1}{cos2\theta} = -\sec 2\theta$$

79)

$$12\cos\theta, 5 + 2\sin\theta, 6\tan\theta$$

i.
we can build an equation based on knowing the sum of the ratio of successive terms must be the same

$$\frac{5+2\sin\theta}{12\cos\theta} = \frac{6\tan\theta}{5+2\,sin\theta}$$

Now we solve

$$(5+2\sin\theta)(5+2\sin\theta) = 12\cos\theta(6\tan\theta)$$
$$25 + 20\sin\theta + 4\sin^2\theta = 72\cos\theta\frac{\sin\theta}{\cos\theta}$$
$$4\sin^2\theta + 20\sin\theta + 25 = 72\sin\theta$$
$$4\sin^2\theta - 52\sin\theta + 25 = 0$$

ii.

let $y = \sin\theta$

$$4y^2 - 52y + 25 = 0$$
$$(2y-1)(2y-25) = 0$$
$$y = \frac{1}{2} y = 12.5$$
$$\sin\theta = \frac{1}{2}, \sin\theta \neq 12.5$$
$$\theta = 30, °, 150°, \ldots$$

Given that θ is obtuse

$$\theta = 150°$$

iii.

The sequence becomes

$$12\cos 150°, 5 + 2\sin 150°, 6\tan 150°$$

We can simplify each term

$$-6\sqrt{3}, 6, -2\sqrt{3}$$

$a = -6\sqrt{3}$

$r = \frac{6}{-6\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$$S_\infty = \frac{a}{1-r} = \frac{-6\sqrt{3}}{1 - -\frac{\sqrt{3}}{3}}$$

Let's multiply ALL terms by 3 to kill the fraction

$$-\frac{18\sqrt{3}}{3+\sqrt{3}}$$

Now we rationalize

$$-\frac{18\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3+-\sqrt{3}}$$

$$= \frac{-54\sqrt{3}+54}{9-3}$$

$$= \frac{-54\sqrt{3}+54}{6}$$

$$= -9\sqrt{3} + 9$$

$$9(1-\sqrt{3})$$

$$k = 9$$

3.2.7 With Sectors

80)

There are two ways to approach this. First, we let the smallest sector have angle θ, and let the circular plank have radius r, so the area is πr^2. We also recall that the area of a sector in terms of its angle, θ, measured in radians is $\frac{r^2\theta}{2}$.

Method 1	**Method 2**
Let A be the smallest sector $(= u_1)$ of the progression, and let the common difference be d. We have $u_{12} = 2u_1$.	Let A be the smallest sector $(= u_1)$ of the progression, and let the common difference be d.

$A + 11d = 2A$ $d = \frac{A}{11}$ Now, we have the sum of all 12 sectors, s_{12}, is equal to the area of the circle. $s_{12} = \pi r^2$ $\frac{12}{2}(2A + (12-1)d) = \pi r^2$ $12A + 66d = \pi r^2$ Substitute in $d = \frac{A}{11}$ $12A + 66\frac{A}{11} = \pi r^2$ $18A = \pi r^2$ $A = \frac{\pi r^2}{18}$ The smallest plank has area $A = \frac{r^2\theta}{2}$ $\frac{r^2\theta}{2} = \frac{\pi r^2}{18}$ $\theta = \frac{\pi}{9}$	The smallest sector has area $\frac{r^2\theta}{2}$, we're told that the largest plank (area u_{12}) has double the area of the smallest plank, so it has area $r^2\theta$. So, $A = \frac{r^2\theta}{2}$, $u_{12} = A + 11d = r^2\theta$, substituting the former into the latter. $\frac{r^2\theta}{2} + 11d = r^2\theta$ $d = \frac{r^2\theta}{22}$ The sum of all 12 sectors is equal to the area of the circle. $s_{12} = \pi r^2$ $\frac{12}{2}(2A + (12-1)d) = \pi r^2$ $12A + 66d = \pi r^2$ Substitute $d = \frac{r^2\theta}{22}$, $A = \frac{r^2\theta}{2}$ $6r^2\theta + 3r^2\theta = \pi r^2$ $9r^2\theta = \pi r^2$ $9\theta = \pi$ $\theta = \frac{\pi}{9}$

81)

Let a sector have angle θ, then it has area $\frac{r^2\theta}{2}$, where r is the radius of the circle.

If another sector (of the same radius) has angle 2θ, then it has area $\frac{r^2(2\theta)}{2} = r^2\theta$. Its area is double that of the smaller circle.

So, if a sector has angle that is twice the angle of another sector, then it has area that is twice of that sector. We can conclude that this question is exactly the same as the previous question. The size of the angle of the smallest sector is $\frac{\pi}{9}$.

82)

OA= 1

By trigonometry, we have $\cos\theta = \frac{adj}{hyp} = \frac{OB}{OA}$

We also have

$\cos\theta = \frac{adj}{hyp} = \frac{OB_1}{OA}$

$\cos\theta = \frac{OB_1}{1}$

$OB_1 = \cos\theta$

We can generalize this to $OB_{n+1} = \cos\theta\,(OA_n)$, for all positive integers n.

Furthermore, since they are both radii of a sector, we have $OA_n = OB_n$, so $OB_{n+1} = \cos\theta\,(OB_n)$. This can be equivalently written as $OB_n = \cos^{n-1}\theta\,(OB_1) = \cos^{n-1}\theta\cos\theta = \cos^n\theta$.

Using the formula of arc length $r\theta$, each arc A_iB_i has length $(OB_i)\theta$.

The sum of arc lengths can now be written as the series

$$(OB)\theta + (OB_1)\theta + (OB_2)\theta + (OB_3)\theta + \cdots$$

We can apply our knowledge $OB_n = \cos^n\theta$, also we use the fact that OB is the radius, which is 1.

$$\theta + \theta\cos\theta + \theta\cos^2\theta + \theta\cos^3\theta + \cdots$$

This is a geometric series with $a = 1, r = \theta$. If it is the case that $-1 < \cos\theta < 1$, then we can substitute this into the formula for sum to infinity.

$$\frac{a}{1-r} = \frac{\theta}{1-\cos\theta}$$

4 Diamond

4.1 Using Formulae

4.1.1 Simultaneous Equations

83)

The first sequence has $a = 15, d = 19 - 15 = 4$. The second sequence has $a' = 420, d' = 415 - 420 = -5$. Let's equate the arithmetic series formula and solve for n.

$$\frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(2a' + (n-1)d')$$

Substitute our values for a, d, a', d'.

$$\frac{n}{2}(30 + 4(n-1)) = \frac{n}{2}(840 - 5(n-1))$$

We can divide both sides by $\frac{n}{2}$, $n = 0$ is a fair assumption to make here.

$$30 + 4(n-1) = 840 - 5(n-1)$$
$$30 + 4n - 4 = 840 - 5n + 5$$
$$9n = 819$$
$$n = 91$$

84)

We have $a = 2k + 1, d = 4k + 4 - (2k + 1) = 4k + 4 - 2k - 1 = 2k + 3$.

Substitute this into the formula $u_n = a + (n-1)d$

$$u_{50} = 2k + 1 + (50 - 1)(2k + 3)$$
$$u_{50} = 2k + 1 + 49(2k + 3)$$
$$u_{50} = 2k + 1 + 98k + 147$$
$$u_{50} = 100k + 148$$

85)

We have $a = b^2 - 13, r = \frac{1}{b}$. Let's substitute these into the sum to infinity formula.

$$-6 = \frac{a}{1-r}$$
$$-6 = \frac{b^2 - 13}{1 - \frac{1}{b}}$$
$$-6 + \frac{6}{b} = b^2 - 13$$
$$-6b + 6 = b^3 - 13b$$
$$b^3 - 7b - 6 = 0$$

Trial and error gives us $b = -1$ as a solution. We now need to factorise the cubic. Set

$$b^3 - 7b - 6 \equiv (b+1)(Ab^2 + Bb + C)$$
$$b^3 + 0b^2 - 7b - 6 \equiv Ab^3 + (A+B)b^2 + (B+C)b + C$$
$$A = 1$$
$$A + B = 0 \Rightarrow B = -1$$
$$C = -6$$

We have

$$b^3 - 7b - 6 = (b+1)(b^2 - b - 6) = 0$$
$$(b+1)(b-3)(b+2) = 0$$

So, the solutions are $b = -1, 3 \text{ } or -2$.

But the sum to infinity formula requires that $-1 < b < 1$.

So, only $b = 3 \text{ } or -2$ are valid. The possible common ratios are $\frac{1}{3}$ and $-\frac{1}{2}$.

86)

i.

We have $a = k, d = 2k - k = k$.

We want to find the number of terms in the series, so we want to find which term has value 100. We solve for $u_n = 100$, using the formula.

$$k + (n-1)k = 100$$
$$k + kn - k = 100$$
$$kn = 100$$
$$n = \frac{100}{k}$$

ii.

We substitute the number of terms $n = \frac{100}{k}$, the first term $a = k$, and the last term $l = 100$ into the formula $s_n = \frac{n}{2}(a + l)$

$$\frac{\frac{100}{k}}{2}(k + 100) = \frac{50}{k}(k + 100) = 50 + \frac{100}{k}$$

87)

We have $a = u_1 = 5t + 3, u_n = 17t + 11, d = 4$, we can solve for n in terms of t.

$$u_n = a + (n-1)d$$
$$17t + 11 = 5t + 3 + (n-1)(4)$$
$$17t + 11 = 5t + 3 + 4n - 4$$
$$4n = 12t + 12$$
$$n = 3t + 3$$

Now, we can use the formula $s_n = \frac{n}{2}(a + l)$ to work out the sum of the series.

$$s_n = \frac{3t+3}{2}(5t + 3 + 17t + 11)$$
$$s_n = \frac{3t+3}{2}(22t + 14)$$
$$s_n = (3t+3)(11t+7)$$

Now, if t is an odd number, $t = 2k + 1$ for some integer k.

$$s_n = (3(2k+1) + 3)(11(2k+3) + 7)$$
$$= (6k + 6)(22k + 40)$$
$$= \big(6(k+1)\big)\big(2(11k + 20)\big)$$
$$= 12(k+1)(11k+20)$$

This is divisible by 12.

If t is an even number, $t = 2k$ for some integer k.

$$s_n = (3(2k)+3)(11(2k)+7)$$
$$= (6k+3)(22k+7)$$

Method 1. Using a parity argument	Method 2. Expand
Both brackets are odd numbers, as they are each an even number plus an odd number. The result is an odd number times an odd number, which is odd. But odd numbers are not divisible by 12. So this number is not divisible by 12.	$s_n = 132k^2 + 108k + 21$ $= 132k^2 + 108k + 12 + 9$ $= 12(11k^2 + 9k + 1) + 9$ This is a multiple of 12 plus 9, which cannot be a multiple of 12.

4.1.2 Arithmetic and Geometric Together in one question

88)

Hint: This is a mix of geometric and arithmetic think of 4,7,10 separately and then $2k, 4k, 8k$. Arithmetic is $3n - 2$ and geometric is 2^{n-1}k.

i.

We have $u_2 = S_2 - S_1, u_3 = S_3 - S_2, u_4 = S_4 - S_3$.

$$u_2 = 5 + 3k - 1 - k = 4 + 2k$$
$$u_3 = 12 + 7k - 5 - 3k = 7 + 4k$$
$$u_4 = 22 + 15k - 12 - 7k = 10 + 8k$$

i.

If we look at the sequence u_1, u_2, u_3, u_4.

$$1 + k, 4 + 2k, 7 + 4k, 10 + 8k$$

We can see that the constants form an arithmetic sequence with initial term $a = 1$, common difference $d = 3$. The terms containing k form a geometric sequence with initial term $b = k$, common ratio $r = 2$.

Then the arithmetic sequence has n^{th} term $1 + (n-1)(3) = 3n - 2$.
The geometric sequence has n^{th}term $k2^{n-1}$

So, the overall n^{th} term is $u_n = 3n - 2 + 2^{n-1}k$.

89)

We let the arithmetic sequence have initial term a, common difference d, and we let the geometric sequence have initial term b, common ratio r.

We can solve for a, d using our knowledge of the sums.

$$24 = s_3 = \frac{3}{2}(2a + (3-1)d)$$
$$24 = 3a + 3d$$

$$55 = s_5 = \frac{5}{2}(2a + (5-1)d$$
$$55 = 5a + 10d$$

Solving the above simultaneously gives us $a = 5, d = 3$.

So, the 3rd, 14th and 58th terms of the arithmetic sequence are

$$5 + 2(3), 5 + 13(3), 5 + 57(3)$$
$$11, 44, 176$$

We have $a = 11, r = \frac{44}{11} = 4$. Let's plug these values into the formula to find the sum of first 5 terms.

$$\frac{11(4^5 - 1)}{4 - 1} = \frac{11(1023)}{3} = 3751$$

90)

Hint: let the three numbers be $x, x + r, x + 2r$ and then they become $x - 1, x + r - 2, x + 2r$

We let the three numbers be $x, x+r, x+2r$, since they are in an arithmetic sequence. They sum to 24, so

$$x+x+r+x+2r=24$$
$$3x+3r=24$$
$$x+r=8$$
$$x=8-r$$

We also have that $x-1, x+r-2, x+2r$ is in a geometric sequence.

$$\frac{x+r-2}{x-1}=\frac{x+2r}{x+r-2}$$
$$(x+r-2)^2=(x+2r)(x-1)$$
$$x^2+r^2+4+2rx-4x-4r=x^2+2rx-x-2r$$
$$r^2+4-4x-4r=-x-2r$$
$$r^2-2r+4-3x=0$$

We can substitute $x=8-r$ into the quadratic.

$$r^2-2r+4-3(8-r)=0$$
$$r^2+r-20=0$$
$$(r+5)(r-4)=0$$
$$r=-5 \text{ or } 4$$

We substitute the values of r back into $x=8-r$. We get

$$x=13, r=-5 \text{ or } x=4, r=4$$

Substitute these back into the sequence

$$13, 8, 3 \text{ or } 4, 8, 12$$

91)

We let the common difference of the arithmetic sequence be d, let the common ratio of the geometric sequence be r.

We have $g_3=a_2$

$$(1+\sqrt{5})r^2=1+\sqrt{5}+d \quad ①$$

We also have $g_4+a_3=0$

$$(1+\sqrt{5})r^3+1+\sqrt{5}+2d=0 \quad ②$$

We can solve these two equations simultaneously. Rearrange ① to get

$$d=(1+\sqrt{5})r^2-1-\sqrt{5}$$

Substitute this into ②

$$(1+\sqrt{5})r^3+1+\sqrt{5}+2\left((1+\sqrt{5})r^2-1-\sqrt{5}\right)=0$$
$$(1+\sqrt{5})r^3+1+\sqrt{5}+2(1+\sqrt{5})r^2-2-2\sqrt{5}=0$$
$$(1+\sqrt{5})r^3+(2+2\sqrt{5})r^2-1-\sqrt{5}=0$$

Divide both sides by $1+\sqrt{5}$

$$r^3+2r^2-1=0$$

Trial and error gives us $r=-1$ as a solution. We now need to factorise the cubic. Set

$$r^3+2r^2-1\equiv(r+1)(Ar^2+Br+C)$$
$$r^3+2r^2+0r-1\equiv Ar^3+(A+B)r^2+(B+C)r+C$$
$$A=1$$
$$A+B=2 \Rightarrow B=1$$
$$C=-1$$

We have

$$r^3+2r^2-1=(r+1)(r^2+r-r)$$

Using the quadratic formula to solve the remaining two roots, we have $r=-1, \frac{-1-\sqrt{5}}{2}$ or $\frac{-1+\sqrt{5}}{2}$.

Since the geometric sequence has a sum to infinity, $-1<r<1$. So $r=\frac{-1+\sqrt{5}}{2}$.

Plugging into the sum to infinity formula.

$$\frac{a}{1-r}=\frac{1+\sqrt{5}}{1-\frac{-1+\sqrt{5}}{2}}=\frac{1+\sqrt{5}}{\frac{3-\sqrt{5}}{2}}=\frac{2+2\sqrt{5}}{3-\sqrt{5}}=\frac{(2+2\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{16+8\sqrt{5}}{9-5}=4+2\sqrt{5}$$

92)

Let the arithmetic sequence have first term a, common difference d. Let the geometric sequence have the first term b, common ratio r. We have the common difference is four time the first term of the geometric sequence.

$$d = 4b$$

The common ratio is twice the first term of the arithmetic sequence.

$$r = 2a$$

If we write out the formula for the first two terms of the sequence

$$\frac{3}{8} = a + b \text{ (1)}$$
$$\frac{13}{16} = a + d + br$$

Substitute d, r.

$$\frac{13}{16} = a + 4b + 2ab \text{ (2)}$$

We can rearrange (1) to $a = \frac{3}{8} - b$, and substitute into (2).

$$\frac{13}{16} = \frac{3}{8} - b + 4b + 2\left(\frac{3}{8} - b\right)b$$
$$\frac{13}{16} = \frac{3}{8} - b + 4b + \frac{3}{4}b - 2b^2$$
$$2b^2 - \frac{15}{4}b + \frac{7}{16} = 0$$
$$32b^2 - 60b + 7 = 0$$
$$b = \frac{1}{8} \text{ or } \frac{7}{4}$$

4.2 With Binomial Expansion

93)

Using binomial expansion, we have

$$(1+kx)^n = 1 + nkx + \frac{n(n-1)}{2}k^2x^2 + \cdots + \frac{n(n-1)(n-2)(n-3)}{24}k^4x^4 + \cdots$$

We have $nk, \frac{n(n-1)}{2}k^2, \frac{n(n-1)(n-2)(n-3)}{24!}k^4$ are consecutive terms of a geometric sequence. They must have the same common ratio.

$$\frac{\frac{n(n-1)}{2}k^2}{nk} = \frac{\frac{n(n-1)(n-2)(n-3)}{24}k^4}{\frac{n(n-1)}{2}k^2}$$

Let's cancel some things out.

$$\frac{(n-1)k}{2} = \frac{(n-2)(n-3)k^2}{12}$$

Divide both sides by $k > 0$, and multiplying out the denominators.

$$12(n-1) = 2(n-2)(n-3)k$$
$$k = \frac{12(n-1)}{2(n-2)(n-3)} = \frac{6(n-1)}{(n-2)(n-3)}$$

4.3 With Logs

94)

We have $a = \frac{1}{\log_2 x}$,

$$d = \frac{1}{\log_8 x} - \frac{1}{\log_2 x}$$

We can use the formula $\log_b x = \frac{\log_c x}{\log_c b}$ to change the log to base 2.

$$d = \frac{1}{\frac{\log_2 x}{\log_2 8}} - \frac{1}{\log_2 x}$$
$$= \frac{\log_2 8}{\log_2 x} - \frac{1}{\log_2 x}$$
$$= \frac{3}{\log_2 x} - \frac{1}{\log_2 x}$$
$$= \frac{2}{\log_2 x}$$

We plug these values into $s_{20} = 100$, and solve for x.

$$\frac{20}{2}\left(2\left(\frac{1}{\log_2 x}\right) + (20-1)\left(\frac{2}{\log_2 x}\right)\right) = 100$$
$$\frac{2}{\log_2 x} + \frac{38}{\log_2 x} = 10$$
$$\frac{40}{\log_2 x} = 10$$
$$4 = \log_2 x$$
$$x = 16$$